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STATISTICAL INFERENCE FOR A SIMPLE STEP–STRESS MODEL WITH TYPE–II HYBRID CENSORED DATA FROM THE KUMARASWAMY WEIBULL DISTRIBUTION

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ABSTRACT

In reliability and life–testing experiments, the researcher is often interested in the effects of extreme or varying stress factors on the lifetimes of experimental units. In this paper, a step–stress model is considered in which the life–testing experiment gets terminated either at a pre–fixed time (say, T_{m+1}) or at a random time ensuring at least a specified number of failures (Say, y out of n). Under this model in which the data obtained are Type–II hybrid censored, the Kumaraswamy Weibull distribution is used for the underlying lifetimes. The maximum Likelihood estimators (MLEs) of the parameters assuming a cumulative exposure model are derived. The confidence intervals of the parameters are also obtained. The hazard rate and reliability functions are estimated at usual conditions of stress. Monte Carlo simulation is carried out to investigate the precision of the maximum likelihood estimates. An application using real data is used to indicate the properties of the maximum likelihood estimators.

Keywords: Accelerated testing; Coverage probability; Cumulative exposure model; Kumaraswamy Weibull distribution; maximum likelihood estimation; Step–stress model; Type–II hybrid censored; Monte Carlo simulation.

1. Introduction

In order to obtain highly reliable products long life spans, consuming and expensive tests are often required to collect enough failure data. The standard life testing methods are not appropriate in such situations, and to overcome this difficulty accelerated life tests are applied; where in the test units are run at higher stress levels (which includes temperature, voltage, pressure, vibration, cycling rate and etc.) to cause rapid failures. Accelerated life tests (ALT) allow the experimenter to apply sever stresses to obtain information on the parameters of the lifetime distributions more quickly than under normal operating conditions. Such tests can reduce the testing time and save a lot of man power, material sources and money. The stress can be applied in different ways: commonly used methods are constant stress, progressive stress and step–stress [Nelson (1990), Bagdonavicius and Nikulin (2002)].

In step–stress ALT, the stress for survival units is generally changed to a higher stress level at a predetermined time. This model assumes that the remaining life of a unit depends only on the current cumulative fraction failed and current stress [Lydersen and Rausand (1987)]. Moreover, if it is held at the current stress, survivors will continue failing according to the cumulative distribution function (CDF) of that stress but starting at the age corresponding to previous fraction failed. This model is called the Cumulative Exposure (CE) model. Some references in the failed of the accelerated life testing include [Jones (2009)]. See also for more details Hassan et al. (2016) and Chunfang and Yimin (2017).

Constructed a distribution with two shape parameters on (0,1). Kumaraswamy (KUM) distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as heights of individuals; scores obtain in a test, atmospheric temperatures and hydro logical data. Also, Kum distribution could be appropriate in situations where scientists use probability distribution which has infinite lower and or upper bounds to fit data, when in reality the bounds are finite. A compound between Kum distribution and any distribution was constructed by [Cordeiro, et. al. (2010)].

Weibull distribution is one of the most popular models; it has been extensively used for modeling data in reliability, engineering and biological studies. The need for forms of Weibull distribution arises in many applied areas. In this paper, simple step–stress is applied to Kumaraswamy Weibull (KUMW) distribution. The cumulative distribution function (CDF) and probability density function (pdf) of KUMW distribution are obtained as follows;

$$F(x;\theta,\beta,\varphi,\lambda)=1-\left[1-\left(1-e^{-(\lambda x)^{\varphi}}\right)^{\theta}\right]^{\beta} \quad , \quad x>0 \tag{1}$$

and,

$$f(x;\theta,\beta,\varphi,\lambda)=\theta\beta\varphi(\lambda)^{\varphi}(x)^{\varphi-1}e^{(-\lambda x)^{\varphi}}\left[1-e^{-(\lambda x)^{\varphi}}\right]^{\theta-1}\left[1-\left(1-e^{-(\lambda x)^{\varphi}}\right)^{\theta}\right]^{\beta-1} \tag{2}$$

$$x>0 \quad , \quad \theta,\beta,\gamma,\varphi>0$$

where, θ,β and φ are the shape parameters, γ is the scale parameter.

It has three shape parameters, these parameters allows for highly degree of flexibility. Some special cases can be obtained from KUMW distribution such as KUM exponential, KUM Rayleigh, Weibull and exponential. It is wide applicable in reliability, engineering and in other areas of research. For more details about the KUMW distribution see [AL–Dayian, et. al. (2014)].

The reliability function (Rf) and the hazard rate function (Hrf) corresponding to (1) can be written, respectively as follows;

$$R(x;\theta,\beta,\varphi,\lambda)=1-\left[1-\left(e^{-(\lambda x)^{\varphi}}\right)^{\theta}\right]^{\beta} \quad , \quad x>0 \tag{3}$$

and,

$$H(x;\theta,\beta,\varphi,\lambda)=\frac{\theta\beta\varphi(\lambda)^{\varphi}(x)^{\varphi-1}e^{(-\lambda x)^{\varphi}}.\left[1-e^{-(\lambda x)^{\varphi}}\right]^{\theta-1}}{\left[1-\left(1-e^{-(\lambda x)^{\varphi}}\right)^{\theta}\right]} \quad , \quad x>0 \tag{4}$$

In this article, a step–stress model is considered in which the life testing experiment gets terminated either at a prefixed time (say, T_{m+1}) or at random time ensuring at least a specified number of failures (say, r out of n). Under this model in which the data obtained are Type–II hybrid censored, the case of two stress levels is proposed with underlying lifetimes being *KUMW* distributed. The model considered here is discussed in detail in section (2). The statistical inference for simple step–stress life testing based on type–II hybrid censored is obtained in section (3).

2. Model Description

Assuming *K* step–stress *ALT*, the model of constant stress is considered in the first step. In this model, the lifetime of the unit is affected by a certain level of stress S_1, where S_1 is larger than the usual stress S_0. In the consecutive steps, other stresses are considered as $S_2, S_3,...,S_k$, where $x_0<x_1<x_2<...<x_k$, then the cumulative exposure model reflects the effect of moving from one stress to another one. In the following subsection some basic assumptions are considered [Trujillo, and Bonat (1989)].

2.1 Basic Assumptions:

1. For any stress S_j, $j=1,2,...,k$, the pdf of the *KUMW* $(\theta,\beta,\varphi,\lambda)$ distribution can be written as follows;

$$f(x_{ij};\theta_j,\beta,\varphi,\lambda)=\theta_j\,\beta\varphi(\lambda)^{\varphi}(x_{ij})^{\varphi-1}e^{(-\lambda x_{ij})^{\varphi}}\left[1-e^{-(\lambda x_{ij})^{\varphi}}\right]^{\theta-1}\left[1-\left(1-e^{-(\lambda x_{ij})^{\varphi}}\right)^{\theta}\right]^{\beta-1} \tag{5}$$

where x_{ij} is a random variable of time at the step j and λ_j is the number of failures at step j, $t_{ij}>0$, $\theta,\beta,\lambda,\varphi>0$, $j=1,2,...,k$ and $i=1,2,...,\lambda_j$.

2. β,φ,γ are constants with respect to the stress S and the shape θ parameter is affected by the stress S_j, $j=1,2,...,k$, through the inverse Power–Law in the form;

$$\theta_j=c\,S_j^{b} \tag{6}$$

where c and b are unknown parameters depending on the nature of the unit and test method.

3. Suppose that, for a particular pattern of stress, units run at stress S_j starting at time τ_{j-1} and reaching to time τ_j , $j=1,2,...,k$ $(\tau_0=0)$. The behavior of such units is as follows;

At step 1: the population fraction $F_1(x)$ of units failing by time τ_1 under constant stress S_1 is;

$$F_1(x)=1-\left[1-\left(1-e^{-(\lambda x_{ij})^{\varphi}}\right)^{c.S_1^b}\right]^{\beta} \qquad ,0<x<\tau_1 \quad , \quad a,b,\beta,\lambda,\varphi>0 \tag{7}$$

If $G(x)$ is the population cumulative distribution on fraction of units failing under step–stress, then in the first step;

$$G(x)=F_1(x) \qquad , \quad 0<x<\tau_1 \tag{8}$$

where τ_1 is the time when the stress is raised from S_1 to S_2.

where step 2: starts, units have equivalent age U_1, which have produced the same fraction failed seen at the end of step 1. In other words the survivors at time τ_1 will be switched to the stress S_2 beginning at the point U_1 which can be determined as the solution of [Hassan and AL–Thobety (2012)];

$$F_2(U_1)=F_1(\tau_1)$$

$$\left[1-\left[1-e^{-(\lambda U_{j-1})^{\varphi}}\right]^{c.S_j^b}\right]^{\beta}=\left[1-\left[1-e^{-\left[\lambda.(\Delta_{j-2}+U_{j-2})\right]^{\varphi}}\right]^{c.S_{j-1}^b}\right]^{\beta} \tag{9}$$

where $\Delta_0=\tau_1-\tau_0$, $U_0=0$ and $\Delta_{j-2}=\tau_{j-1}-\tau_{j-2}$, $j=2,3,...,k$, by solving (9), one obtain;

$$e^{(-\lambda U_{j-1})^{\varphi}}=1-\left[1-e^{\left(-\lambda.(\Delta_{j-2}+U_{j-2})\right)^{\varphi}}\right]^{c.\left(\frac{S_{j-1}}{S_j}\right)^b}$$

By taking the logarithm for two sides, it follows that;

$$U_{j-1}=\frac{1}{\lambda}.\left\{-\ell n\left[1-\left(1-e^{\left(-\lambda.(\Delta_{j-2}+U_{j-2})\right)^{\varphi}}\right)^{b.\left(\frac{S_{j-1}}{S_j}\right)^b}\right]\right\}^{\frac{1}{\varphi}} \tag{10}$$

The cumulative exposure model for j steps can be written as follows;

$$G(x)=F_1(x-\tau_{j-1}+U_{j-1}) \qquad , \quad \tau_{j-1}<x<\tau_j$$

$$G(x)=1-\left[1-\left(1-e^{-\left[\lambda.(x-\tau_{j-1}+U_{j-1})\right]^{\varphi}}\right)^{c.S_{j-1}^b}\right]^{\beta} \tag{11}$$

Substituting U_{j-1} in (11), it is seen that $G(x)$, for a step–stress pattern which consists of segments of the *CDF*, $F_1, F_2, ..., F_k$, can be written in the form;

$$G(x)=\begin{cases} 0 & , \quad x \le \tau_0 \\ F_1(x) & , \quad \tau_0 \le x \le \tau_1 \\ \\ F_j(x-\tau_{j-1}+U_{j-1}) & , \quad \tau_{j-1} \le x \le \tau_j \quad , \quad j=2,3,...,k-1 \\ F_k(x-\tau_{k-1}+U_{k-1}) & , \quad \tau_{k-1} \le x \le \infty \end{cases} \tag{12}$$

and the associated *pdf*, $g(x)$, has the following form;

$$g(x)=\begin{cases} f_1(x) & , \quad \tau_0 \le x \le \tau_1 \\ f_j(x-\tau_{j-1}+U_{j-1}) & , \quad \tau_{j-1} \le x \le \tau_j \quad , \quad j=2,3,...,k-1 \\ \\ f_k(x-\tau_{k-1}+U_{k-1}) & , \quad \tau_{k-1} \le x \le \infty \\ zero & , \quad otherwise \end{cases} \tag{13}$$

2.1 Cumulative Exposure Model:

The life time distribution at S_1 and S_2 from (12) and (13); $G(x)$ and $g(x)$, can be written in the form;

$$G(x_{ij})=\begin{cases} G_1(x_{i1})=1-\left[1-\left(1-e^{(-\lambda x_{i1})^{\varphi}}\right)^{c.S_1^b}\right]^{\beta} & , \quad 0<x<\tau_1 \\ \\ G_2(x_{i2})=1-\left[1-\left\{1-e^{\left[-\lambda(x_{i2}-\tau_1+U_1)\right]^{\varphi}}\right\}^{c.S_2^b}\right]^{\beta} & , \quad \tau_1<x<\tau_2 \end{cases} \tag{14}$$

and the associated density function $g(x)$ is written in the form;

$$g(x_{ij})=\begin{cases} g_1(x_{i1})=\varphi\beta\lambda^{\varphi}c\,S_1^b\,(x_{i1})^{\varphi-1}e^{-(\lambda x_{i1})^{\varphi}}.\left[1-e^{-(\lambda x_{i1})^{\varphi}}\right]^{c.S_1^b-1}.\left[1-\left(1-e^{-(\lambda x_{i1})^{\varphi}}\right)^{c.S_1^b}\right] \\ \qquad , \quad \tau_0 \le x \le \tau_1 \\ g_2(x_{i2})=\varphi\beta\lambda^{\varphi}c\,S_2^b\left(x_{i2}-\tau_1+U_1\right)^{\varphi-1}.e^{-\lambda(x_{i2}-\tau_1+U_1)^{\varphi}} \\ \quad .\left(1-e^{-\lambda(x_{i2}-\tau_1+U_1)^{\varphi}}\right)^{c.S_2^b-1}\left[1-\left(1-e^{-\lambda(x_{i2}-\tau_1+U_1)^{\varphi}}\right)^{c.S_2^b-1}\right]^{\beta-1} \\ \qquad , \quad \tau_1 \le x \le \tau_2 \end{cases} \tag{15}$$

3. Inference Simple Step–Stress Accelerated Life Tests Based on Type II–Hybrid Censored Samples:

Based on the type–II hybrid censored sample (*HCS*), there is n identical units under an initial stress level S_1. The stress level is changed to S_2 at time τ_1, and the life testing experiment is terminated at random time τ_2^*. Here $\tau_2^* = \max(x_{r,n}, \tau_2)$, where;

(i) $r(\le n)$ and $0 < \tau_1 < \tau_2 < \infty$ are fixed in advance,

(ii) τ_1 denotes a fixed time at which the stress level is changed from S_1 to S_2,

(iii) $x_{1,n} < x_{2,n} < ... < x_{r,n}$ denote the order failure times of n units under test,

(iv) $x_{r,n}$ denotes the time when the r^{th} failure occurs,

(v) τ_2 denotes a fixed time before which if the r^{th} failure occurs the experiment is terminated at time τ_2.

(vi) τ_2^* denotes the random time when the life–testing experiment is terminated [Balakrishnan, and Xie (2007)].

Let,

$N_1 =$ Number of units that fail before time τ_1;

$N_2 =$ Number of units that fail before time τ_2 at stress level S_1; and

$N_2^* =$ Number of units that fail before time τ_2^* at stress level S_1.

Then, it is evident that;

$$N_2^* = \begin{cases} r - N_1 \quad , & if \quad x_{r,n} > \tau_2 \\ \\ N_2 \quad , & if \quad x_{r,n} \le \tau_2 \end{cases} \quad \Leftrightarrow \quad \tau_2^* = \begin{cases} x_{r,n} \quad , & if \quad x_{r,n} > \tau_2 \\ \\ \tau_2 \quad , & if \quad x_{r,n} \le \tau_2 \end{cases} \tag{16}$$

With the notation, we will observe one of the following three cases for the observations:

Case 1: if $x_{r,n} \le \tau_1 < \tau_2$, we will observe that;

$$\left\{x_{(1,n)} < ... < x_{(r,n)} < ... < x_{(N_1,n)} \le \tau_1 < x_{(N_1+1,n)} < ... < x_{(N_1+N_2,n)} \le \tau_2\right\}$$

Case 2: if $\tau_1 < x_{r,n} \le \tau_2$, we will observe that;

$$\left\{x_{(1,n)} < ... < x_{(N_1,n)} < \tau_1 < x_{(N_1+1,n)} < ... < x_{(r,n)} < ... < x_{(N_1+N_2,n)} \le \tau_2\right\}$$

Case 3: if $x_{r,n} > \tau_2$, we will observe that;

$$\left\{x_{(1,n)} < ... < x_{(N_1,n)} \le \tau_1 < x_{(N_1+1,n)} < ... < x_{(N_1+N_2)} < \tau_2 < x_{(N_1+N_2+1,n)} < x_{(r,n)} < \infty\right\}$$

The failure time distribution is assumed to be *KUMW* distribution and the shape parameter θ is shown as a function of the stress through the inverse power law model.

3.1 Maximum Likelihood Estimation:

From the *CED* in (14) and the corresponding *pdf* in (15), the likelihood function of c, b, β, φ and λ based on the type–II *HCS* is obtained as follows;

$$L(c,b,\beta,\varphi,\lambda / x) = \frac{n!}{(n-r^*)!}\left[\prod_{i=1}^{N_1} g_1(x_{1i})\right]\left[\prod_{i=N_1+1}^{r^*} g_2(x_{2i})\right]\left[1-G_2(\tau_2^*)\right]^{n-r^*}$$

$$L=\frac{n!}{(n-r^*)!}\left(\beta.\varphi.c.S_1^b.\lambda^{\varphi}\right)^{N_1}.\left(\beta.\varphi.c.S_2^b.\lambda^{\varphi}\right)^{N_2^*}.\left\{\prod_{i=1}^{N_1}(x_{ij})^{\varphi-1}.e^{(-\lambda x_{ij})^{\varphi}}\left(1-e^{-(\lambda x_{ij})^{\varphi}}\right)^{c.S_1^b-1}\right.$$

$$\left.\left[1-\left(1-e^{-(\lambda x_{ij})^{\varphi}}\right)^{c.S_1^b}\right]^{\beta-1}\right\}.\left\{\prod_{i=N_1+1}^{r^*}\left(x_{ij}-\tau_1+U_1\right)^{\varphi-1}.\left(e^{-\left(\lambda(x_{ij}-\tau_1+U_1\right)^{\varphi}}\right).\right.$$

$$\left.\left(1-e^{-\left(\lambda(x_{ij}-\tau_1+U_1\right)^{\varphi}}\right)^{c.S_2^b-1}.\left(1-\left(1-e^{-\left(\lambda(x_{ij}-\tau_1+U_1\right)^{\varphi}}\right)\right)^{c.S_2^b-1}\right\}$$

$$.\left\{\left(1-\left(1-e^{-\left(\lambda(\tau_2^*-\tau_1+U_1\right)^{\varphi}}\right)\right)^{c.S_2^b-1}\right\}^{\beta(n-r^*)} \tag{17}$$

where; $0<x_{(1,n)}<...<x_{(N_1,n)}\le\tau_1<x_{(N_1+1,n)}<...<x_{(r^*,n)}<\tau_2^*$

In this situation the log likelihood function of c,b,β,φ and λ is given by;

$$\ell n\,L\,\alpha(N_1+N_2^*).\ell n\,(c.\beta.\varphi)+(N_1+N_2^*)\varphi.\ell n(\lambda)+b\sum_{j=1}^{2}N_j.\ell n(S_j)+(\varphi-1)\sum_{i=1}^{N_1}\ell n(x_{ij})-\sum_{i=1}^{N_1}(\lambda x_{ij})^{\varphi}+$$

$$(c.S_1^b-1)\sum_{i=1}^{N_1}\ell n\left(H_1(.)\right)+(\beta-1)\sum_{i=1}^{N_1}\ell n\left(1-H_2(.)\right)+(\varphi-1)\sum_{i=1}^{r^*}\ell n\left(H_3(.)\right)-\sum_{i=1}^{r^*}\ell n\left(\lambda.H_3(.)\right)^{\varphi}+(c.S_2^b-1).$$

$$\sum_{i=1}^{r^*}\ell n\left(1-e^{-(\lambda.H_3(.))^{\varphi}}\right)+(\beta-1)\sum_{i=1}^{r^*}\ell n\left(1-H_4(.)\right)+\beta(n-r^*)\ell n\left(1-H_5(.)\right) \tag{18}$$

where,

$$H_1(.)=\left(1-e^{-(\lambda.x_{ij})^{\varphi}}\right);$$

$$H_2(.)=\left(1-e^{-(\lambda.x_{ij})^{\varphi}}\right)^{c.S_1^b}=\left(H_1(.)\right)^{c.S_1^b};$$

$$H_3(.)=\left(x_{i2}-\tau_1+U_1\right);$$

$$H_4(.)=\left(1-e^{-\left(\lambda(x_{i2}-\tau_1+U_1\right)^{\varphi}}\right)^{c.S_2^b-1}=\left(1-e^{-(\lambda.H_3(.))^{\varphi}}\right)^{c.S_2^b-1};$$

and,

$$H_5(.)=\left(1-e^{-\left(\lambda(x_{i2}-\tau_1+U_1\right)^{\varphi}}\right)^{c.S_2^b-1}=\left(1-e^{-(\lambda.H_3(.))^{\varphi}}\right)^{c.S_2^b-1}$$

The first derivatives of the logarithm of likelihood function (18) with respect to, c,b,β,φ and λ are obtained. Therefore, the *MLE* can be obtained by equating the first derivatives of $\ell n(L)$ to zero. As shown they are nonlinear equations, the estimates $\hat{c},\hat{b},\hat{\beta},\varphi$ and $\hat{\lambda}$ are numerically using Newton Raphson method [Chenhua (2009)]. Depending on the invariance of the *MLEs*, the *MLE* of the shape parameter θ_j, of the *KUMW* distribution at usual stress S_u, can be estimated using the following equation;

$$\hat{\theta}_{2U} = \hat{c}.S_{2U}^{\hat{b}} \quad (19)$$

also, the *MLE* of the *rf* under the same usual conditions can be given by;

$$\hat{R}(\tau_0) = \left[1 - \left(1 - e^{-\left(\hat{\lambda}.\tau_0\right)^{\hat{\varphi}}} \right)^{\hat{\theta}_2.U} \right]^{\hat{\beta}} \quad (20)$$

and, the *MLE* of the *hrf* under the same conditions is given as follows;

$$\hat{H}(\tau_0) = \frac{\hat{\theta}_{2U}.\hat{\beta}.\hat{\varphi}.\hat{\lambda}^{\hat{\varphi}}.\hat{\tau}_0^{\varphi-1}\left(e^{-\left(\hat{\lambda}.\tau_0\right)^{\hat{\varphi}}} \right)\left(1 - e^{-\left(\lambda.\tau_0\right)^{\varphi}} \right)^{\hat{\theta}_{2U}-1}}{\left(1 - \left(1 - e^{-\left(\hat{\lambda}.\tau_0\right)^{\hat{\varphi}}} \right)^{\hat{\theta}_{2U}} \right)} \quad (21)$$

Where, τ_0 is a mission time. The asymptotic Fisher–Information Matrix can be written as follows;

$$I_2 = \left[-\frac{\partial^2 \ell n(L)}{\partial \psi_i \, \partial \psi_j} \right] \quad , \quad i, j = 1,2,3,4,5 \quad (22)$$

Where, $\psi_1 = c$, $\psi_2 = b$, $\psi_3 = \beta$, $\psi_4 = \varphi$, $\psi_5 = \lambda$ and the elements of the information matrix (22) were derived.

3.2 The Confidence Interval (CIs) Based on Type–II HCS:

Based on large sample size, the *MLEs* under appropriate regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed. Therefore, the two side approximate $100(1-\alpha)\%$ confidence intervals for the *MLE* say, $\hat{w}$ of population value w can be obtained by $P\left(-Z \le \frac{\hat{w} - w}{\sigma_{\hat{w}}} \le Z \right) = 1 - \alpha$, where Z is the $100(1-\alpha/2)$ the standard normal percentile. The two sides approximate $100(1-\alpha)\%$ confidence intervals for c, b, β, φ and λ will be respectively, as follows;

$$L_w = \hat{w} - Z_{\alpha/2}.\sigma_w \quad , \quad and \quad U_w = \hat{w} + Z_{\alpha/2}.\sigma_w \quad (23)$$

where $\hat{\sigma}_w$ is the standard deviation and this study $\hat{w}$ is $\hat{c}$ or $\hat{b}$ or $\hat{\beta}$ or $\hat{\varphi}$ and or $\hat{\lambda}$, respectively [Nelson (1982)].

4. Numerical Results

This section aims to investigate the precision of the theoretical results of both estimation and optimal design plans on basis of simulated and real data.

4.1 Simulation Study:

Several data sets are generated from *KUMW* distribution for a combination of the initial parameters values of c, b, β, φ and λ for sample sizes 25, 50, 75, 100 and 125 using 1000 replications for each sample size. The transformation between uniform distribution and *KUMW* distribution in step $j = 1$ is;

$$U_1 = 1 - \left[1 - \left(1 - e^{-(\lambda x_i)^{\varphi}} \right)^{c.S_1^b} \right]^{\beta} \tag{24}$$

While the transformation between uniform distribution and *KUMI* distribution in step $j = 2$ is;

$$U_2 = 1 - \left[1 - \left(1 - e^{-(\lambda S)^{\varphi}} \right)^{c.S_2^b} \right]^{\beta} \tag{25}$$

where,

$$S = \left[x_{i2} - \tau_1 - \left\{ \frac{1}{N\lambda} .\ell n \left(1 - \left(1 - e^{-(\lambda x_{i1})^{\varphi}} \right)^{\left(\frac{S_1}{S_2}\right)^b} \right) \right\} \right]^{\frac{1}{\varphi}} \tag{26}$$

The whole size n is with initial values of parameters $\beta = 0.5, \varphi = 1, \lambda = 2, c = 1$ and $b = 1$ given $N_1 = 0.4n, N_2 = 0.5n$ and $N_c = 0.1n$. It is assumed that there are only two level of stress of $(k = 2)$ $S_1 = 1$ and $S_2 = 2$ which are higher than the stress at usual condition, $S_u = 0.5$. Number of (l) of stress where, $G_1 = 0.4, G_2 = 0.5, r = 40\%(n_j), and\ r = 80\%(n_j), j = 1,2$. The initial parameter values of c, b, β, φ and λ are used in this simulation study to generate x_{ij} , $j = 1,2$ and $i = 1,2,...,r_j^*$.

Computer program is used depending on "Mathcad 14" using Newton Raphson method to solve the derived nonlinear logarithmic likelihood equations simultaneously.

Based on 1000 replications S with $S_1 = 1, S_2 = 2, \tau_1 = 1$ and $\tau_2 = 1.5$ are fixed and different sample sizes and r . Once the values of $\hat{c}, \hat{b}, \hat{\beta}, \varphi$ and $\hat{\lambda}$ are obtained, the estimates are used to obtain, depending on (18) and the design stress, $S_u = 0.5$, the shape parameter under this stress θ_u is estimated as $\hat{\theta}_u = \hat{c}.S_u^{\hat{b}}$. Also, the *rf* and *hrf* are estimated at different values of mission times under usual conditions using (20) and (21).

The performance of $\hat{c}, \hat{b}, \hat{\beta}, \varphi$ and $\hat{\lambda}$ has been evaluated through some measurements of accuracy. In order to study the precision and variation of *MLEs,* it is convenient to use the root mean squares error. The coverage probabilities and length for c, b, β, φ and λ, Root Mean Squares Error (*RMSE*), Absolute Value of Reliability Error (*ARE*) and Relative Bias (*RB*) will be obtained. The different sample size of $n = 25, 50, 75, 100$ and 125 are considered. These sample sizes are chosen to present the small, moderate and large sizes. The results are displayed in tables (1) to (3).

It is clear from table (1) that, the *MLEs* are much closed to the initial values of the parameters as sample size increases. Also, as shown in the numerical results *RMSE, ARE* and *RB* are decreasing when the sample size is increasing. For all sample sizes the following are observed;

- $\hat{c}$, $\hat{b}$ and $\hat{\lambda}$ performs better than $\hat{\beta}$ and $\hat{\varphi}$.
- $\hat{b}$ and $\hat{\lambda}$ performs better than $\hat{c}$.
- $\hat{b}$ performs better than $\hat{c}$.

Table 1: MLEs of Unknown Parameters β,φ,λ,c and b, *RMSE*, *ARE* and *RB* with different Censoring Scheme $\beta=0.5,\varphi=1,\lambda=2,c=1$ and $b=1$

n	*r = n%*	*Parameters*	*MLE*	*RMSE*	*ARE*	*RB*
25	0.4	β	0.615	0.691	1.383	0.23
		φ	1.116	1.278	2.556	1.233
		λ	1.67	1.713	0.857	0.165
		c	1.161	1.581	1.581	0.161
		b	0.893	1.647	1.647	0.107
	0.8	β	0.623	0.716	1.432	0.246
		φ	1.257	1.316	2.631	1.514
		λ	1.906	2.242	1.121	0.047
		c	1.055	1.375	1.375	0.055
		b	0.834	1.565	1.565	0.166
50	0.4	β	0.609	0.682	1.363	0.219
		φ	1.051	1.145	2.29	1.103
		λ	1.618	1.537	0.768	0.191
		c	0.991	1.153	1.153	0.00895
		b	0.867	1.608	1.608	0.133
	0.8	β	0.638	0.707	1.413	0.275
		φ	1.153	1.16	2.32	1.305
		λ	1.889	2.018	1.009	0.056
		c	0.985	1.194	1.194	0.015
		b	0.942	1.703	1.703	0.058
75	0.4	β	0.719	0.773	1.546	0.439
		φ	1.065	1.09	2.181	1.131
		λ	2.061	1.891	0.945	0.03
		c	1.075	1.123	1.123	0.075
		b	1.02	1.799	1.799	0.02
	0.8	β	0.696	0.734	1.467	0.392
		φ	1.016	1.001	2.002	1.033
		λ	1.963	1.916	0.958	0.018
		c	1.046	1.176	1.176	0.046
		b	1.026	1.757	1.757	0.026
100	0.4	β	0.622	0.688	1.376	0.245
		φ	1.025	1.104	2.208	1.05
		λ	1.617	1.492	0.746	0.192
		c	0.98	1.341	1.341	0.02
		b	0.907	1.666	1.666	0.093
	0.8	β	0.702	0.724	1.449	0.404
		φ	1.001	0.979	1.959	1.001
		λ	1.95	1.806	0.903	0.025
		c	1.014	1.069	1.069	0.014
		b	1.078	1.804	1.804	0.078
125	0.4	β	0.597	0.639	1.277	0.194
		φ	0.961	1.033	2.066	0.923
		λ	1.652	1.422	0.711	0.174
		c	0.979	1.006	1.006	0.021
		b	0.809	1.514	1.514	0.191

0.8	β	0.703	0.729	1.458	0.405
	φ	1.062	1.09	2.18	1.123
	λ	1.919	1.838	0.919	0.041
	c	1.01	1.083	1.083	0.01
	b	1.07	1.777	1.777	0.07

Table (2) indicates that; the reliability decreases when the mission time to increases. The results get better in the sense that the aim of an accelerated life testing experiments is to get large number of failure (reduce the reliability) of the device with high reliability. As to and when the sample size increase, the *rf* increase. The *hrf* increases when the mission time to increase.

Table (2): The Reliability Function and the Hazard Rate Function with Different Censoring Scheme

n \ t		0.1	0.5	0.75	1	1.5
25		$(r=40\%n)$				
	$\hat{R}(t)$	0.821	0.482	0.352	0.257	0.136
	$\hat{H}(t)$	7.663	13.047	17.861	24.454	46.271
		$(r=80\%n)$				
	$\hat{R}(t)$	0.814	0.426	0.28	0.18	0.07
	$\hat{H}(t)$	7.551	14.446	21.98	34.132	87.595
50		$(r=40\%n)$				
	$\hat{R}(t)$	0.777	0.457	0.342	0.258	0.148
	$\hat{H}(t)$	7.613	12.932	17.275	22.887	39.841
		$(r=80\%n)$				
	$\hat{R}(t)$	0.752	0.387	0.262	0.177	0.079
	$\hat{H}(t)$	7.587	14.767	21.807	32.284	72.225
75		$(r=40\%n)$				
	$\hat{R}(t)$	0.701	0.323	0.21	0.137	0.058
	$\hat{H}(t)$	7.413	16.056	24.784	37.999	89.265
		$(r=80\%n)$				
	$\hat{R}(t)$	0.695	0.342	0.234	0.162	0.079
	$\hat{H}(t)$	7.458	15.141	22.165	32.027	65.854
100		$(r=40\%n)$				
	$\hat{R}(t)$	0.756	0.439	0.329	0.249	0.145
	$\hat{H}(t)$	7.612	13.1	17.501	23.121	39.801
		$(r=80\%n)$				
	$\hat{R}(t)$	0.67	0.327	0.224	0.156	0.077
	$\hat{H}(t)$	7.482	15.321	22.343	32.076	64.766
125		$(r=40\%n)$				
	$\hat{R}(t)$	0.765	0.461	0.354	0.276	0.171
	$\hat{H}(t)$	7.239	12.011	15.625	20.047	32.303
		$(r=80\%n)$				
	$\hat{R}(t)$	0.688	0.332	0.223	0.151	0.07
	$\hat{H}(t)$	7.61	15.776	23.507	34.695	75.126

The two sides 95% central asymptotic *CIs* for the parameters of *KUMW* are displayed in table (3). This table contains the lower bound (*L*), upper bound (*U*), length of the intervals and the coverage probabilities. The interval estimate of the parameters becomes narrower as the sample size increases. For all sample sizes, it is clear that;

– The length of the interval for β is shorter than the other lengths.

– The length of the interval for c is shorter than the length of the interval b .

– Among *CIs* seems to have considerably low coverage (especially φ) compared to the nominal level. The coverage probabilities better and closer to the nominal level in general.

Table (3): The Confidence Interval Results and the Coverage Probability with Different Censoring Scheme

n	*r =n%*	*LL*	*UL*	*Length*	*Cov*
25	0.4	-0.721	1.951	2.673	91.4
		-1.078	3.311	4.389	89.2
		-1.625	4.965	6.591	97.4
		-1.921	4.965	6.164	97.4
		-1.831	3.618	5.449	93.7
	0.8	-0.76	2.006	2.765	92
		-0.852	3.366	4.219	84.8
		-2.483	6.296	8.779	95
		-1.637	6.296	5.385	95.6
		-1.775	3.443	5.218	94.2
50	0.4	-0.709	1.928	2.638	90.7
		-0.916	3.019	3.935	88
		-1.3	4.535	5.835	96.7
		-1.268	4.535	4.519	97.5
		-1.799	3.533	5.332	92.7
	0.8	-0.721	1.996	2.717	91.1
		-0.726	3.032	3.758	84.6
		-2.06	5.837	7.897	95.3
		-1.356	5.837	4.682	96.2
		-1.851	3.735	5.586	93.7
75	0.4	-0.734	2.172	2.906	90.1
		-0.762	2.893	3.654	87.1
		-1.643	5.765	7.408	95
		-1.121	5.765	4.392	96.1
		-1.9	3.939	5.839	93.1
	0.8	-0.689	2.082	2.771	91.5
		-0.664	2.697	3.361	86.4
		-1.791	5.718	7.509	94.7
		-1.257	5.718	4.605	95.9
		-1.782	3.834	5.617	94.6
100	0.4	-0.705	1.95	2.655	90
		-0.878	2.929	3.807	87.8
		-1.208	4.442	5.65	97
		-1.648	4.442	5.256	98.9
		-1.843	3.658	5.501	92.7
	0.8	-0.661	2.066	2.727	91.1
		-0.649	2.65	3.3	86.3
		-1.588	5.487	7.075	94.1
		-1.08	5.487	4.19	95.6

		-1.771	3.927	5.699	95.2
125	0.4	-0.64	1.834	2.474	91.3
		-0.85	2.773	3.624	87.5
		-1.05	4.355	5.406	96.5
		-0.993	4.355	3.944	97
		-1.713	3.33	5.043	93.5
	0.8	-0.67	2.075	2.745	91.3
		-0.769	2.892	3.661	86.1
		-1.681	5.518	7.199	96.2
		-1.112	5.518	4.245	96.2
		-1.727	3.866	5.593	95.3

4.2 Illustrative Example:

In this subsection, the main aim is to demonstrate how the proposed method can be used in practice. Using Kolomogrov–Smirnov goodness of fit test where the data points representing failure time. The data were taken from [Murthy, et. al. (2004).] and analyzed by [Rezk1et al. (2014)], the data were 20 items (*n=20*) tested with test stopped after 20th failure $(r = 20)$. The Kolomogrov–Smirnov test shows that the *KUMW* distribution provides a good fit to the data. It is assumed that $k = 2$, i.e. there are only two different levels of stresses $S_1 = 1$ and $S_2 = 2$, which are higher than the stress at usual conditions, $S_u = 0.5$. The failure times in the first and the second steps are;

Stress–level	Time–to–failure											
$S_1 = 1$	0.02	0.06	1.38	2.01	2.53	2.82	3.15	4.98	**5.55**	**5.82**	**5.87**	**7.47**
$S_2 = 2$	7.51	7.67	8.61	9.04	9.12	9.65	10.16	10.76				

The initial parameter values of c, b, β, φ and λ used in this application are $\beta = 0.5, \varphi = 0.5, \lambda = 2, c = 1$ and $b = 1$. Once the estimate values of c, b, β, φ and λ are obtained, the estimators are used to estimate θ_u, as $\theta_u = \hat{c} S_u^{\hat{b}}$. Letting the design stress, $S_u = 0.5$. Also, the reliability function is estimated at different values of mission times under conditions depending on (20).

In this case, we had a fixed time $\tau_2 = 10$ and $r = 15, 19$, the *MLEs* of c, b, β, φ and λ from (18) are $\hat{c}$ = 1.06; $\hat{b}$ = 0.98,0.94; $\hat{\beta}$ = 0.65,0.59; $\hat{\phi}$ = 2.98,2.75 and $\hat{\lambda} = 1.93, 1.90$ respectively. Note that when $r = 15$, we have $\tau_2^* = \max\{x_{15,30}, \tau_2\} = \max\{8.61, 10\} = 10$, Similarly, when $r = 18$, we find $\tau_2^* = 10.16$ (Type–II HSC).

5. Summary and Conclusions

In this paper, a simple step–stress model with two stress levels from *KUMW* distribution is considered where the data are type–II HCS. The *MLEs* of the unknown parameters $\hat{c}, \hat{b}, \hat{\beta}, \varphi$ and $\hat{\lambda}$ are derived. A simulation study is conducted to compare the performance of all these procedures. It is observed that the approximation method of constructing *CIs* (based on the asymptotic normality of *MLEs*, c, b, β, φ and λ) is satisfactory in terms of coverage probabilities (quite close to the nominal level) for the parameters c, b, β, φ and

λ. An example is presented to illustrate all methods of inference discussed here as well as to support the conclusions drawn. As a future work, one can estimate the *KUMW* distribution parameters under simple step–stress model based on some recently introduced schemes like joint progressive Type-I censored scheme, . Abo-Kasem and Nassar (2019).

The results obtained in this paper can be modified to obtain results for sub–models of *KumW* distribution under type–II HCS such as; the *KumW* exponential distribution, if $\varphi = 1$, the *KumW* Rayleigh distribution, if $\varphi = 2$, the exponentiated Wibull distribution, if $\beta = 1$,the exponentiated Rayleigh distribution, if $\beta = 1, \varphi = 2$, the exponentiated exponential distribution, if $\beta = \varphi = 1$, the Wibull distribution, if $\beta = \theta = 1$. [Khamis (1997)], the exponential distribution, if $\varphi = 2, \beta = \theta = 1$, and the Rayleigh distribution, if $\varphi = \beta = \theta = 1$.

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