

STATISTICAL INFERENCE FOR TWO WEIBULL POPULATIONS BASED ON JOINT PROGRESSIVE TYPE-I CENSORED SCHEME

Osama E. Abo-Kasem^{1*}, Mazen M. Nassar²

^{1,2}*Department of Statistics, Faculty of Commerce, Zagazig University, Egypt*

ABSTRACT

In this paper, maximum likelihood and Bayesian methods of estimation are used to estimate the unknown parameters of two Weibull populations with the same shape parameter under joint progressive Type-I (JPT-I) censoring scheme. Bayes estimates of the parameters are obtained based on squared error and LINEX loss functions under the assumption of independent gamma priors. We propose to apply Markov Chain Monte Carlo (MCMC) technique to carry out a Bayesian estimation procedure. The approximate confidence intervals and the credible intervals for the unknown parameters are also obtained. Finally, we analyze a one real data set for illustration purpose.

Keywords: Joint progressive Type-I censored scheme, Weibull distribution, Maximum likelihood estimation, Confidence bounds, Bayesian estimation, Squared-error loss, LINEX loss, Markov Chain Monte Carlo.

* Corresponding author
Email: usama_eraky84@yahoo.com

1. Introduction

Censored sampling arises in a life testing experiment whenever the experimenter does not observe the failure time of all items placed on a life test. There are various types of censored data to be dealt with in the analysis of lifetime experiments see [Lawless, 2003]. Almost all of these types of data are concerned with the one-sample problems. But, there are situations in which the experimenter plans to compare different populations. In such problems, the joint censoring scheme has been suggested in the literature. As mentioned by Balakrishnan and Rasouli (2008), Rasouli and Balakrishnan (2010) and Ashour and Abo-Kasem (2017), a joint censoring scheme is quite useful in conducting comparative lifetime test of products coming from different units within the same facility.

More precisely, suppose that the products are being produced by two lines under the same facility. Two independent samples of sizes m and n are selected from these lines and put simultaneously on a life testing experiment. Then, to save time and money, the experimenter follows a joint censoring scheme and terminates the life testing when a certain number of failures (say, r) occur. In the literature, Balakrishnan and Rasouli (2008) developed likelihood inference for the parameters of two exponential populations under joint Type-II censoring. Shafay et al. (2013), Ashour and Abo-Kasem (2014a), Ashour and Abo-Kasem (2014b), Ashour and Abo-Kasem (2014c) and Balakrishnan and Feng (2015) considered a jointly Type-II censored sample for some lifetime distributions. Rasouli and Balakrishnan (2010) extended their work to the case of two exponential populations when joint progressive Type-II censoring is implemented on the two samples. See also. Parsi et al. (2011), Doostparast et al. (2013) and Balakrishnan et al. (2015).

Recently, Ashour and Abo-Kasem (2017) introduced JPT-I censored and as a special case, joint Type-I censored scheme. Joint progressive Type-I censored is carried out as: Consider m identical items from product A are put on a test and the lifetimes of m items are denoted by X_1, \dots, X_m , and n identical items from product B are put on a test and the lifetimes of n items are denoted by Y_1, \dots, Y_n . Further, suppose $W_{(1)} \leq W_{(2)} \leq \dots \leq W_{(N)}$ denote the order statistics of the $N = m + n$ random variables $\{X_1, \dots, X_m; Y_1, \dots, Y_n\}$. Now, a JPT-I scheme between the two samples is implemented as follows. At a predetermined time T_1 , R_1 units are randomly withdrawn from the remaining $N - r_1$ (r_1 number of units failed in time interval $(T_0 - T_1)$ and they belongs to X and \ or Y) surviving units. Next, at the second predetermined time T_2 , R_2 units are randomly withdrawn from the remaining $N - R_1 - r_1 - r_2$ (r_2 number of units failed in time interval $(T_1 - T_2)$ and they belongs to X and \ or Y) surviving units, and so on. Finally, at the predetermined time T_k all remaining $R_k = N - \sum_{i=1}^k r_i - \sum_{i=1}^{k-1} R_i$ surviving units are withdrawn from the life-testing experiment. The total number of complete failures $r = \sum_{i=1}^k r_i$ as well as the Type-I progressive censoring scheme

(R_1, R_2, \dots, R_k) are prefixed. Suppose $R_j = s_j + q_j, j = 1, \dots, k$, where s_j and q_j are the number of units withdrawn at the time of the j th failure that belongs to X and Y samples, and these are unknown and random variables. The observed data in this form will consist of (Z, W, S) where $W = (W_1, \dots, W_r)$ with $r < N$ being a prefixed integer, $Z = (Z_1, \dots, Z_r)$ with $Z_i = 1$ or 0 according as whether W_i is either an X or an Y failure, respectively, and $S = (S_1, \dots, S_k)$. Of course, as mentioned above, the progressive Type-I censoring scheme $R = (R_1, \dots, R_k)$ has the decomposition $S + Q = (S_1, \dots, S_k) + (Q_1, \dots, Q_k)$.

Ashour and Abo-Kasem (2017) obtained Bayesian and non-Bayesian estimators for two exponential populations under both JPT-I censored and joint Type-I censored. Due to the limitations of the exponential distribution in modeling lifetime data because its constant failure rate, we consider inference of the parameters of two Weibull populations under JPT-I scheme. The importance of using Weibull distribution over the exponential distribution comes from its popularity and flexibility in modeling lifetime data with increasing or decreasing failure rate. The main aim of this paper is to investigate the estimation problems of two Weibull populations with the same shape parameter under JPT-I censoring scheme. We obtain the maximum likelihood estimators (MLEs) and the approximate confidence intervals. We also obtain the Bayes estimates using squared error and LINEX loss functions under the assumption of independent gamma priors. It is known that when the shape parameter is known, the scale parameter has a conjugate gamma prior, but when the shape parameter is unknown the conjugate priors do not exist. To obtain the Bayes estimates and the credible intervals in this case we assume that the shape parameter has gamma prior distribution.

The rest of this paper is organized as follows: The maximum likelihood estimation and the approximate confidence intervals are obtained in Section 2. In Section 3, we discuss Bayesian estimation under squared error and LINEX loss functions. In Section 4, a numerical example is considered to illustrate the proposed estimators and the analysis of real data set is presented. Last Section includes a brief conclusion.

2. Maximum Likelihood Estimation

Using JPC-I scheme; Ashour and Abo-Kasem (2017) derived the likelihood of $(Z, W, \text{ and } S)$ as follows

$$L = C \prod_{i=1}^r [\{f(w_i)\}^{Z_i} \{g(w_i)\}^{1-Z_i}] \prod_{j=1}^k [\{\bar{F}(T_j)\}^{S_j} \{\bar{G}(T_j)\}^{q_j}] \quad (1)$$

where C is a constant does not depend on the parameters, $M_r = \sum_{i=1}^r Z_i$ denote the number of X -failures in W and $N_r = \sum_{i=1}^r (1 - Z_i) = r - M_r$ (i.e., the number of Y -failures in W), $\bar{F} = 1 - F, \bar{G} = 1 - G$ are the survival functions of the two populations, $\sum_{j=1}^k S_j + \sum_{j=1}^k q_j = \sum_{j=1}^k R_j, \sum_{j=1}^k S_j = m - m_r$ and $\sum_{j=1}^k q_j = n - n_r$.

In this paper, we assume that the two populations are Weibull distributions with the same shape parameter (β) and different scale parameters ($\lambda_\delta, \delta = 1,2$) with density and distribution functions as $\beta, \lambda_\delta > 0, x > 0, F_\delta = 1 - \exp(-\lambda_\delta \lambda_\delta x^\beta)$ and $f_\delta = \beta \lambda_\delta x^{\beta-1} \exp(-\lambda_\delta x^\beta)$ for $\delta = 1,2$, respectively. In this case, the likelihood function in (1) becomes

$$L = \beta^r \lambda_1^{m_r} \lambda_2^{n_r} e^{-\lambda_1 \psi_1(w_i, \beta)} e^{-\lambda_2 \psi_2(w_i, \beta)} \prod_{i=1}^r w_i^{\beta-1} \tag{2}$$

where $\psi_1(w_i, \beta) = \sum_{i=1}^r z_i w_i^\beta + \sum_{j=1}^k s_j T_j^\beta$ and $\psi_2(w_i, \beta) = \sum_{i=1}^r (1 - z_i) w_i^\beta + \sum_{j=1}^k (R_j - s_j) T_j^\beta$. The log-likelihood function $l = \ln L$ is

$$l = r \ln \beta + m_r \ln \lambda_1 + n_r \ln \lambda_2 + (\beta - 1) \sum_{i=1}^r \ln w_i - \lambda_1 \psi_1(w_i, \beta) - \lambda_2 \psi_2(w_i, \beta) \tag{3}$$

when the common shape parameter β is known, we can obtain the MLEs of λ_1 and λ_2 after taking the derivation of (3) with respect to λ_1 and λ_2 and equating the results to zero as

$$\hat{\lambda}_1 = \frac{m_r}{\psi_1(w_i, \beta)} \text{ and } \hat{\lambda}_2 = \frac{n_r}{\psi_2(w_i, \beta)} \tag{4}$$

Now we consider the case when the common shape parameter is unknown, which is most likely to happen in practice. For fixed β , the MLEs of λ_1 and λ_2 can be obtained as

$$\hat{\lambda}_1(\beta) = \frac{m_r}{\psi_1(w_i, \beta)} \text{ and } \hat{\lambda}_2(\beta) = \frac{n_r}{\psi_2(w_i, \beta)} \tag{5}$$

Substituting $\hat{\lambda}_1(\beta)$ and $\hat{\lambda}_2(\beta)$ in (3), we can obtain the profile log-likelihood of as follows

$$p(\beta) = r \ln \beta + \beta \sum_{i=1}^r \ln w_i - m_r \ln[\psi_1(w_i, \beta)] - n_r [\psi_2(w_i, \beta)] \tag{6}$$

The MLE of β can be obtained by maximizing (6) with respect to β . Since $p(\beta)$ is unimodal [see Pareek et al. (2009)], the MLE of β say $\hat{\beta}$ can be obtained by differentiate $p(\beta)$ with respect to β then equating the result by zero, we will have the estimator of β as $\beta = h(\beta)$, where

$$h(\beta) = \left[\frac{m_r \gamma_1(w_i, \beta)}{r \psi_1(w_i, \beta)} + \frac{n_r \gamma_2(w_i, \beta)}{r \psi_2(w_i, \beta)} - \frac{\sum_{i=1}^r \ln(w_i)}{r} \right]^{-1} \tag{7}$$

where $\gamma_1 = \sum_{i=1}^r z_i w_i^\beta \ln w_i + \sum_{j=1}^k s_j T_j^\beta \ln T_j$ and $\gamma_2 = \sum_{i=1}^r (1 - z_i) w_i^\beta \ln w_i + \sum_{j=1}^k (R_j - s_j) T_j^\beta \ln T_j$

Most of the standard iterative process can be used to find the MLE of β , a simple iterative scheme proposed by Pareek et al. (2009) can be used. Once we obtain (5) from obtained be can λ_2 and λ_1 of the MLE the, $\hat{\beta}$.

Remark: From the 4 and 7, it is to be noted that when $M_r = \sum_{i=1}^r z_i = 0$ or r , $\hat{\lambda}_1$ or $\hat{\lambda}_2$ and $\hat{\beta}$ do not exist, respectively. Hence, the MLEs in 4 and 7 are conditioned on $1 \leq M_r \leq r - 1$.

To construct confidence intervals for the unknown parameters we need to compute the asymptotic variance-covariance matrix which obtained by inverting the Fisher information matrix $I(\lambda_1, \lambda_2, \gamma)$, in which elements are negatives of expected values of the second partial derivatives of l . The elements of the sample information matrix will be

$$I_{11} = -\frac{\partial^2 l}{\partial \lambda_1^2} = \frac{m_r}{\lambda_1^2},$$

$$I_{22} = -\frac{\partial^2 l}{\partial \lambda_2^2} = \frac{n_r}{\lambda_2^2},$$

$$I_{33} = -\frac{\partial^2 l}{\partial \beta^2} = \frac{r}{\beta^2} + \lambda_1 \left[\sum_{i=1}^r z_i w_i^\beta (\ln w_i)^2 + \sum_{j=1}^k s_j T_j^\beta (\ln T_j)^2 \right]$$

$$+ \lambda_2 \left[\sum_{i=1}^r (1 - z_i) w_i^\beta (\ln w_i)^2 + \sum_{j=1}^k (R_j - s_j) T_j^\beta (\ln T_j)^2 \right],$$

$$I_{12} = I_{21} = -\frac{\partial^2 l}{\partial \lambda_1 \partial \lambda_2} = 0$$

$$I_{13} = I_{31} = -\frac{\partial^2 l}{\partial \lambda_1 \partial \beta} = \gamma_1$$

and

$$I_{23} = I_{32} = -\frac{\partial^2 l}{\partial \lambda_2 \partial \beta} = \gamma_2$$

Under some regularity conditions, $(\hat{\lambda}_1, \lambda_2, \hat{\beta})$ is approximately normal with mean $(\lambda_1, \lambda_2, \beta)$ and covariance matrix $I^{-1}(\lambda_1, \lambda_2, \beta)$. Practically, we estimate $I^{-1}(\lambda_1, \lambda_2, \beta)$ by $I^{-1}(\hat{\lambda}_1, \lambda_2, \hat{\beta})$, then

$$I^{-1}(\hat{\lambda}_1, \lambda_2, \hat{\beta}) = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}_{(\hat{\lambda}_1, \lambda_2, \hat{\beta})}^{-1} = \begin{bmatrix} \text{Var}(\hat{\lambda}_1) & \text{cov}(\lambda_1, \hat{\lambda}_2) & \text{cov}(\lambda_1, \hat{\beta}) \\ \text{cov}(\lambda_1, \hat{\lambda}_2) & \text{Var}(\lambda_2) & \text{cov}(\hat{\lambda}_2, \beta) \\ \text{cov}(\hat{\lambda}_1, \beta) & \text{cov}(\hat{\lambda}_2, \beta) & \text{Var}(\hat{\beta}) \end{bmatrix}$$

Now, the approximate confidence intervals for λ_1 , λ_2 and β can be obtained as follow

$$\hat{\lambda}_\delta \pm z_{1-\frac{\alpha}{2}} \sqrt{\text{Var}(\lambda_\delta)}, \delta = 1, 2 \text{ and } \hat{\beta} \pm z_{1-\frac{\alpha}{2}} \sqrt{\text{Var}(\beta)}$$

where z_q is the $100q$ - th percentile of a standard normal distribution.

3. Bayesian Estimation

In this section, we discuss the Bayesian estimates and the corresponding credible intervals in two cases; the first case when the common shape parameter is known, the second when it is unknown.

3.1 Common shape parameter β is known

When the common shape parameter β is known, we use the following joint prior density of λ_1 and λ_2

$$g(\lambda_1, \lambda_2) \propto \lambda_1^{\alpha_1-1} \lambda_2^{\alpha_2-1} e^{-[\lambda_1 b_1 + \lambda_2 b_2]}, \quad a_1, a_2, b_1, b_2 > 0 \quad (8)$$

it is to be observed that $\lambda_1 \sim \text{Gamma}(a_1, b_1)$ and $\lambda_2 \sim \text{Gamma}(a_2, b_2)$. It follows from (2) and (8) that the joint posterior of λ_1 and λ_2 is

$$L(\lambda_1, \lambda_2 | \beta, \mathbf{x}) \propto \lambda_1^{m_r + \alpha_1 - 1} \lambda_2^{n_r + \alpha_2 - 1} e^{-\lambda_1 (b_1 + \varphi_1(w_i, \beta))} e^{-\lambda_2 (b_2 + \varphi_2(w_i, \beta))} \quad (9)$$

From (9), we can conclude the following

$$\begin{aligned} \lambda_1, \text{Gamma}(m_r + a_1, b_1 + \varphi_1(w_i, \beta)) \sim. \\ \lambda_2, \text{Gamma}(n_r + a_2, b_2 + \varphi_2(w_i, \beta)) \sim. \end{aligned}$$

To obtain the Bayesian estimates, we consider two types of loss functions, namely, squared error (SE) loss function and LINEX loss function introduced by Varian (1975). The LINEX loss function takes the form

$$l(\tilde{\theta}, \theta) = c\{e^{v(\tilde{\theta}-\theta)} - v(\tilde{\theta} - \theta) - 1\} \quad (10)$$

where c and v are constants. From (10) the Bayes estimate of θ , denoted by $\tilde{\theta}$ is given by

$$\tilde{\theta}_{LIN} = -\frac{1}{v} \ln E(e^{-v\theta}), \quad v \neq 0 \quad (11)$$

From (9) and (11), the Bayes estimates of λ_1 and λ_2 under SE loss function are obtained as follow

$$\tilde{\lambda}_{SE1} = \frac{m_r + a_1}{b_1 + \varphi_1(w_i, \beta)} \quad \text{and} \quad \tilde{\lambda}_{SE2} = \frac{n_r + a_2}{b_2 + \varphi_2(w_i, \beta)}$$

It is seen that when $a_1 = b_1 = a_2 = b_2 = 0$, the Bayes estimates coincide with the MLEs of λ_1 and λ_2 given by (4). Now, it can be easily to prove that $\varphi_1 = 2\lambda_1(b_1 + \varphi_1(w_i, \beta))$ and $\varphi_2 = 2\lambda_2(b_2 + \varphi_2(w_i, \beta))$ follow x^2 distributions with degrees of freedom $[2(m_r + a_1)]$ and $[2(n_r + a_2)]$, respectively. Therefore, $100(1 - \alpha)\%$ credible intervals for λ_1 and λ_2 are

$$\left\{ \frac{x^2_{[2(m_r+a_1)], 1-\frac{\alpha}{2}}}{2(b_1 + \varphi_1(w_i, \beta))}, \frac{x^2_{[2(m_r+a_1)], \frac{\alpha}{2}}}{2(b_1 + \varphi_1(w_i, \beta))} \right\} \text{ and } \left\{ \frac{x^2_{[2(n_r+a_2)], 1-\frac{\alpha}{2}}}{2(b_2 + \varphi_2(w_i, \beta))}, \frac{x^2_{[2(n_r+a_2)], \frac{\alpha}{2}}}{2(b_2 + \varphi_2(w_i, \beta))} \right\}$$

On the other hand, the gamma distribution can be used to obtain these credible intervals when $[2(m_r + a_1)]$ and $[2(n_r + a_2)]$ are not integer values.

Using (10), the Bayes estimates of λ_1 and λ_2 under LINEX loss function can be obtained as follows

$$\tilde{\lambda}_{LIN1} = -\frac{m_r + a_1}{v} \ln \left\{ \frac{b_1 + \varphi_1(w_i, \beta)}{v + b_1 + \varphi_1(w_i, \beta)} \right\}$$

and

$$v \neq 0, \tilde{\lambda}_{LIN2} = -\frac{n_r + a_2}{v} \left\{ \frac{b_2 + \varphi_2(w_i, \beta)}{v + b_2 + \varphi_2(w_i, \beta)} \right\}$$

3.2 Common shape parameter β is unknown

Here, we assume that λ_1 and λ_2 have gamma priors with joint prior density given in (8), and β follows Gamma(c,d). The joint prior of λ_1 , λ_2 and β is given by

$$g(\lambda_1, \lambda_2, \beta) \propto \lambda_1^{\alpha_1-1} \lambda_2^{\alpha_2-1} \beta^{c-1} e^{-[\lambda_1 b_1 + \lambda_2 b_2 + \beta d]}, a_1, a_2, b_1, b_2, c, d > 0 \quad (12)$$

from (2) and (12), the joint posterior of λ_1 , λ_2 and β is

$$L(\lambda_1, \lambda_2, \beta, x) \propto$$

$$\beta^{r+c-1} \lambda_1^{m_r+a_1-1} \lambda_2^{n_r+a_2-1} e^{-\lambda_1(b_1+\varphi_1(w_i,\beta))} e^{-\lambda_2(b_2+\varphi_2(w_i,\beta))} e^{-d\beta} \times \prod_{i=1}^r w_i^{\beta-1}$$

For any function, say $U(\lambda_1, \lambda_2, \beta)$, the Bayes estimate of $U(\lambda_1, \lambda_2, \beta)$ is

$$\tilde{U}(\lambda_1, \lambda_2, \beta) = \frac{\int_0^\infty \int_0^\infty \int_0^\infty U(\lambda_1, \lambda_2, \beta) L(\lambda_1, \lambda_2, \beta) g(\lambda_1, \lambda_2, \beta) d\lambda_1 d\lambda_2 d\beta}{\int_0^\infty \int_0^\infty \int_0^\infty L(\lambda_1, \lambda_2, \beta) g(\lambda_1, \lambda_2, \beta) d\lambda_1 d\lambda_2 d\beta} \quad (13)$$

Since (13) is very complicated and cannot be computed analytically, MCMC technique can be used to approximate (13). We apply MCMC technique to generate samples from (12) and use these samples to compute the Bayes estimates and the corresponding credible intervals. Before applying MCMC technique, we need the following results

- $\lambda_1 | \lambda_2, \beta, x \sim \text{Gamma}(m_r + a_1, b_1 + \varphi_1(w_i, \beta))$,
- $\lambda_2 | \lambda_1, \beta, x \sim \text{Gamma}(n_r + a_2, b_2 + \varphi_2(w_i, \beta))$,
- $g(\beta | \lambda_1, \lambda_2, x) \propto \beta^{r+c-1} e^{-\beta d} e^{-\lambda_1 \varphi_1(w_i, \beta)} e^{-\lambda_2 \varphi_2(w_i, \beta)} \times \prod_{i=1}^r w_i^{\beta-1}$

It is to be noted that $g(\beta | \lambda_1, \lambda_2, x)$, is not known, but Kundu (2008) proved that $g(\beta | x)$ is a log-concave function in the form

$$g(\beta | x) \propto \beta^{r+c-1} e^{-\beta d} \frac{[b_1 + \varphi_1(w_i, \beta)]^{-(m_r+a_1)}}{[b_2 + \varphi_2(w_i, \beta)]^{n_r+a_2}} \times \prod_{i=1}^r w_i^{\beta-1} \quad (14)$$

Because λ_1 and λ_2 are gamma densities, samples of λ_1 and λ_2 can be easily generated. However, the function in (14) cannot be reduced to well-known distribution, but we observed by plotting it that it is similar to normal distribution, so we use the normal distribution as an approximation to $g(\beta | x)$, see Fig(1). To obtain the Bayes estimates and the corresponding credible intervals, we use the following MCMC steps

Step 1: Set $t = 1$.

Step 2: Generate $\beta^{(t)}$ from (14) with proposal distribution $q(\beta) = N(\hat{\beta}, \text{var}(\beta))$ as follows:

- a) Let $\varepsilon = \beta^{(t-1)}$. We use the MLE of β as $\beta^{(0)}$.
- b) Generate ω from $N(\hat{\beta}, \text{var}(\beta))$
- c) Accept ω with probability $P(\varepsilon, \omega) = \min(1, \frac{q(\varepsilon)g(\omega|x)}{q(\omega)g(\varepsilon|x)})$.

Step 3: Generate $\lambda_1^{(t)}$ and from $\lambda_2^{(t)}$ and $\text{Gamma}(m_r + a_1, b_1 + \varphi_1(w_i, \beta))$, respectively $\text{Gamma}(n_r + a_2, b_2 + \varphi_2(w_i, \beta))$.

Step 4: Set $t = t + 1$.

Step 5: Repeat steps 2-4, M times, and obtain $(\beta^{(i)}, \lambda_1^{(i)}, \lambda_2^{(i)})$, $i = 1, \dots, M$.

Step 6: Obtain the Bayes estimates of β , λ_1 and λ_2 under SE loss function as

$$\tilde{\theta}_{SE} = \frac{\sum_{i=1}^M \theta^{(i)}}{M}, \theta = \beta, \lambda_1, \lambda_2$$

Step 7: Obtain the $100(1 - 2\alpha)\%$ symmetric credible intervals of β , λ_1 and λ_2 as $(\theta^{[\alpha M]}, \theta^{[(1-\alpha)M]})$

where $\theta^{[i]}$ is the ascending order of $\theta^{(i)}$, $\theta = \beta, \lambda_1, \lambda_2$.

Step 8: Obtain the Bayes estimates of λ_1 , λ_2 and β under LINEX loss function as

$$\tilde{\theta}_{LIN} = -\frac{1}{v} \ln(\sum_{i=1}^M e^{-v\theta^{(i)}} / M), \theta = \beta, \lambda_1, \lambda_2 .$$

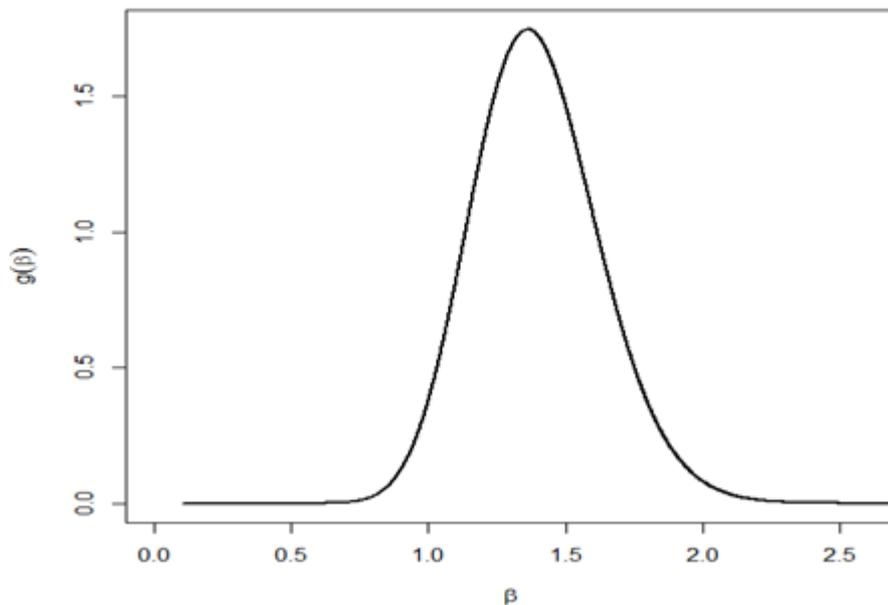


Figure (1). Posterior density function of β .

4. Numerical Illustration

The main object of this section is to illustrate numerically the new theoretical result obtained in the previous two sections.

Example (1): Real data-set

To illustrate the usefulness of the proposed estimators obtained in sections 2 and 3 with real situations, we consider two samples of size $m = n = 10$ each from Nelson's data (1982), (groups 3 and 5 in Table 4.1, [27, p.462]) which correspond to breakdown in minutes of an insulating fluid subjected to high voltage stress. These failure times, denoted here as groups X and Y, are presented in table 1.

Table 1: The failure time data for groups X and Y.

Group	Data									
X	1.99,	0.64,	2.15,	1.08,	2.57,	0.93,	4.75,	0.82,	2.06,	0.49
Y	8.11,	3.17,	5.55,	0.80,	0.20,	1.13,	6.63,	1.08,	2.44,	0.78

Table 2 presents the JPT-I censored sample that have been obtained from the two samples in table 1. The JPT-I censored sample are chosen by considering $(R_1 = 2, s_1 = 1)$, $T_3 = 3$ and $T_2 = 2, T_1 = 1$ (in minutes). Furthermore, we use $(R_3 = 3, s_3 = 1)$, and $(R_2 = 2, s_1 = 1)$. All of the computations were performed using MATHCAD program version 2007.

Table 2: The JPT-I data from Nelson's data (1982), (groups 3 and 5 in Table 4.1, [27, p.462])

J	N_j	Failure Times w_i	z_i	r_i	R_j	s_j	q_j
1	20	0.2,0.49,0.64,0.78,0.8,0.82,0.93	0,1,1,0,0,1,1	7	2	1	1
2	11	1.08,1.13,1.99	0,0,1	3	2	1	1
3	6	2.06,2.44,2.57	1,0,1	3	3	1	2
Total				13	7	3	4

Before progressive further, we plot the profile log-likelihood function (6) in Fig (2). From Fig (2), it is noted that the profile log-likelihood function is unimodal, therefore we propose to use the initial value of $\beta = 1.5$ to start the iteration to obtain the MLE of β . The MLEs are $\hat{\beta} = 1.40687$, $\hat{\lambda}_1 = 0.3487$ and $\hat{\lambda}_2 = 0.2944$, and the corresponding 95% confidence intervals are (0.766,2.473), (0.049,0.648) and (0.021,0.5677), respectively. We assume the noninformative priors to obtain the Bayes estimates because we have no prior information about the parameters. We can use the gamma density function as an approximation to the posterior density function of β by equating the first two moments. From the data, the shape

and scale parameters of the gamma density function are 18.359 and 13.16, respectively. Based on the MCMC samples of size 10000, the Bayes estimates under SE loss function are $\tilde{\beta}_{SE} = 1.4937$, $\tilde{\lambda}_{SE1} = 0.3504$ and $\tilde{\lambda}_{SE2} = 0.2952$, while the credible intervals are (1.1137,2.0534), (0.144,0.6551) and (0.1108,0.5729), respectively. The Bayes estimates under LINEX loss function with $v = 1.5$ are $\tilde{\beta}_{LIN} = 1.44874$, $\tilde{\lambda}_{LIN1} = 0.3379$ and $\tilde{\lambda}_{LIN2} = 0.2848$.

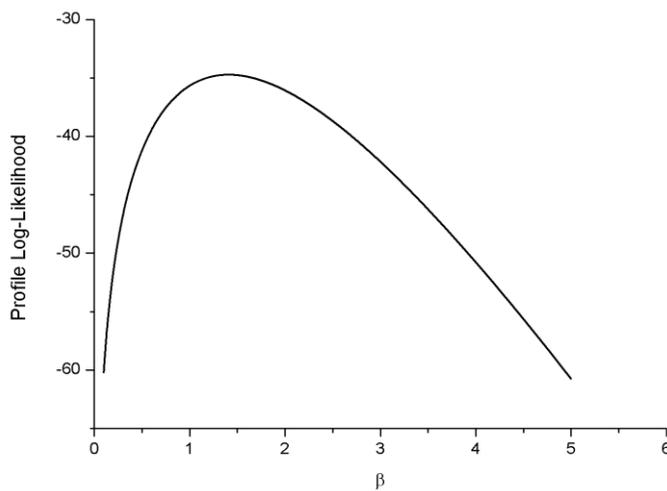


Figure (2). Profile log-likelihood function of β .

Example (2) Simulated example

To illustrate the use of the estimation method proposed in this article, a JPT-I censored sample are generated from two Weibull populations with parameters $\lambda_1 = 0.5, \lambda_2 = 0.6$ and $\beta = 2$ and $m = n = 20$ with the following censoring schemes:

Scheme1: $(s_j = 4,1,1)$ and $k = 3, (T_j = 0.75,1.25,1.75), (R_j = 6,3,4)$ where $(j = 1,2,3)$

Scheme2: $(s_j = 3,2,1,1,2)$ and $k = 5, (T_j = 0.5,0.75,1.25,1.5,1.75), (R_j = 4,3,2,4,4)$ where $(j = 1,2, \dots, 5)$. The generated data are displayed in table 3 for scheme 1 and scheme 2.

Table 3: The JPT-I censored data from scheme 1 and 2

Scheme 1									
J	N_j	Failure Times	w_i	z_i	r_i	R_j	s_j	q_j	
1	40	0.05,0.121,0.134,0.343,0.438,0.504,0.5		1,0,1,1,1,1,	12	6	4	2	
		65,0.603,0.619,0.655,0.718,0.733		1,1,1,1,0,0					
2	22	0.851,0.886,0.929,1.01,1.017,1.052,		1,0,1,0,0,0,	10	3	1	2	
		1.095,1.137,1.216,1,232		1,0,0,1					
3	9	1.255,1.373,1.508,1.575,1.588		0,0,0,1,0	5	4	1	3	
Total					27	13	6	7	
Scheme 2									
1	40	0.05, 0.121, 0.134, 0.343, 0.438		1, 0, 1, 1, 1	5	4	3	1	
2	31	0.504,0.619,0.718,0.733		1, 1, 0, 0	4	3	2	1	
3	24	1.017, 1.095, 1.137, 1.211, 1.216, 1.232		0, 1, 0, 1, 0, 1	6	2	1	1	
4	16	1.255, 1.357, 1.373		0, 1, 0	3	4	1	3	
5	9	1.506, 1.508, 1.588, 1.741, 1.749		0, 0, 0, 0, 1	5	4	2	2	
Total					23	17	9	8	

We then obtain the MLEs, mean squared error (MSE), and the Bayes estimates by considering two types of priors, Prior 0: $a_1 = b_1 = a_2 = b_2 = c = d = 0$ and Prior 1: $a_1 = 1, b_1 = 2, a_2 = 0.6, b_2 = 1, c = 4, d = 2$. Note that, Prior 0 is chosen to describe the non-informative prior case, while prior 1 is selected in such way that prior means are same as the original means. The MLEs and Bayes estimates under SE and LINEX (with $v = 0.5$) loss functions of λ_1, λ_2 and β based on the data in table 3 are presented in table 4. Table 5 presents the 95% approximate confidence and Bayes credible intervals for λ_1, λ_2 and β .

Table 4: The MLEs, Bayesian estimates and MSE's within brackets of $\lambda_1 = 0.5, \lambda_2 = 0.6$ and $\beta = 2$ using different schemes

Schemes	MLEs	Bayesian estimates				
		Squared error loss		LINEX loss ($v = 0.5$)		
		Prior 0	Prior 1	Prior 0	Prior 1	
1	$\hat{\lambda}_1$	0.902 (0.22)	0.909 (0.167)	0.855 (0.126)	0.894 (0.155)	0.845 (0.119)
	$\hat{\lambda}_2$	0.463 (0.037)	0.477 (0.015)	0.466 (0.018)	0.473 (0.016)	0.464 (0.018)
	$\hat{\beta}$	1.830 (0.117)	1.945 (0.003)	2.043 (0.0018)	1.938 (0.004)	1.881 (0.014)
2	$\hat{\lambda}_1$	0.542 (0.03)	0.522(0.0005)	0.545 (0.002)	0.515(0.0002)	0.539(0.0015)
	$\hat{\lambda}_2$	0.394 (0.057)	0.377(0.0499)	0.409 (0.0363)	0.374 (0.051)	0.406 (0.038)
	$\hat{\beta}$	1.713 (0.179)	1.986 (0.0002)	1.957 (0.0019)	1.976 (0.001)	1.951 (0.002)

Table 5: The 95% approximate and Bayes credible confidence intervals for λ_1 , λ_2 and β using different schemes

CI	Scheme (1)			Scheme (2)		
	λ_1	λ_2	β	λ_1	λ_2	β
Approximate						
	(0.428, 1.377)	(0.2, 0.726)	(1.249,2.412)	(0.212,0.87)	(0.157,0.63)	(1.105,2.322)
Bayes credible						
Prior 0	(0.476,1.39)	(0.231,0.693)	(1.769,2.648)	(0.258,0.87)	(0.198,0.612)	(1.641,2.451)
Prior 1	(0.477, 1.325)	(0.252,0.742)	(1.766,2.436)	(0.277,0.89)	(0.213,0.669)	(1.679,2.305)

From table 4 and 5, it is observed that the Bayes estimates perform better than MLEs in terms of minimum MSE and confidence length. Comparing the two priors, we can see that the Bayes estimates under Prior 1 perform better than those based on Prior 0 for both the two schemes. Also, it is noted that the estimates of λ_1 and β have better performance in scheme 2 than scheme 1 while the estimate of λ_2 is better in scheme 1 than scheme 2. Finally, it can be seen that the Bayes estimates using LINEX loss function are performing better than those based on squared error loss for both two schemes.

5. Conclusions

In this article, we have consider the maximum likelihood and Bayesian estimation for the unknown parameters of two Weibull distributions with the same shape parameter based on a JPT-I scheme. We obtained the MLEs of the parameters as well as the corresponding approximate confidence intervals. The Bayes estimates and the credible intervals are obtained using the assumption of independent gamma priors under squared error and LINEX loss functions. We propose to apply MCMC technique to carry out a Bayesian estimation procedure. Finally, we analyze a real data set and simulated example. As a future work, the inferential results discussed in this paper can be performed for some lifetime distributions as the Exponentiated Burr XII Weibull distribution and the Exponentiated Weibull-Lomax Distribution under a JPT-I scheme.

References

- [1] Ashour S. K and Abo-Kasem O. E. (2014a). Bayesian and non-Bayesian estimation for two generalized exponential populations under joint type-II censored scheme. *Pakistan Journal of Statistics and Operation Research*. 10 (1), 57- 72.
- [2] Ashour S. K and Abo-Kasem O. E. (2014b). Parameter estimation for two Weibull populations under joint type-II censored scheme. *International Journal of Engineering and Applied Sciences*. 5 (4), 31-36.
- [3] Ashour S. K and Abo-Kasem O. E. (2014c). Parameter estimation for multiple Weibull populations under joint type-II censoring. *International Journal of Advanced Statistics and Probability*. 2 (2), 102-107.
- [4] Ashour S. K and Abo-Kasem O. E. (2017). Statistical inference for two exponential populations under joint progressive type-I censored scheme. *Communications in Statistics-Theory and Methods*. 46(7), 3479-3488.
- [5] Balakrishnan, N. and Rasouli, A. (2008). Exact likelihood inference for two exponential populations under joint type-II censoring. *Computational Statistics & Data Analysis*, 52, 2725 – 2738.
- [6] Balakrishnan N. and Feng S. (2015). Exact likelihood inference for k exponential populations under joint type-II censoring. *Communications in Statistics - Simulation and Computation*. 44(3), 591-613.
- [7] Balakrishnan N., Feng S. and Kin-yat L.(2015). Exact likelihood inference for k exponential populations under joint progressive type-II censoring. *Communications in Statistics - Simulation and Computation*. 44(3), 902-923.
- [8] Doostparast, M., Ahmadi, M., Vali and Ahmadi, J. (2013). Bayes estimation based on joint progressive type-II censored data under LINEX loss function. *Communications in Statistics - Simulation and Computation*. 42(8), 1865-1886.
- [9] Hassan, A. S. and Abd-Allah, M. (2018). Exponentiated Weibull-Lomax Distribution: Properties and Estimation. *Journal of Data Science*. 16(2), 277-298.
- [10] Kundu, D. (2008). Bayesian inference and life testing plan for the Weibull distribution in presence of progressive censoring. *Technometrics*. 50:144–154.

- [11] Lawless, J. F. (2003). *Statistical Models and Methods for Life Time Data*. 2nd Edition, John Wiley, New York.
- [12] Makubatea, B., Dingaloo, N., Oluyedeb, B. O. and Fagbamigbea, A.F. (2018). The Exponentiated Burr XII Weibull Distribution: Model, Properties and Applications. *Journal of Data Science*. 16(3), 431-462.
- [13] Nelson W. (1982). *Applied life data analysis*. New York: Wiley.
- [14] Pareek, B., Kundu, D. and Kumar, S. (2009). On progressively censored competing risks data for Weibull distribution. *Computational Statistics and Data Analysis*, 53, 4083-4094.
- [15] Parsi, S. Ganjali, M. and Sanjari Farsipour N. (2011). Conditional maximum likelihood and interval estimation for two Weibull populations under joint type-II progressive censoring. *Communications in Statistics-Theory and Methods*. (40) 2117-2135.
- [16] Rasouli, A., and Balakrishnan, N. (2010). Exact likelihood inference for two exponential populations under joint progressive type-II censoring. *Communications in Statistics-Theory and Methods*. 39 (12), 2172–2191.
- [17] Shafay, A. R., Balakrishnan, N. and Abdel-Aty, Y. (2013). Bayesian inference based on a jointly type-II censored sample from two exponential populations. *Communications in Statistics - Simulation and Computation*. 43, 1-14.
- [18] Varian, H. R. (1975). A Bayesian approach to real state assessment. In: Stephen, E. F. & Zellner, A. (Eds.) *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*, 195–208. Amsterdam: North-Holland.