

THE COMPOUND CLASS OF JANARDAN-POWER SERIES DISTRIBUTIONS: PROPERTIES AND APPLICATIONS

Marzieh Shekari*, Hossein Zamani*, Mohammad Mehdi Saber²

**Department of Statistics, Hormozgan University, Bandar Abbas, Iran.*

²Department of Statistics, College of Sciences, Higher Education Center of Eghlid, Eghlid, Iran.

Abstract

In the present paper, we propose the new Janardan-Power Series (JPS) class of distributions, which is a result of combining the Janardan distribution of Shanker et.al (2013) with the family of power series distributions. Here, we examine the fundamental attributes of this class of distribution, including the survival, hazard and reverse hazard functions, limiting behavior of the cdf and pdf, quantile function, moments and distribution of order statistics. Moreover, the particular case of the JPS distribution such as the Janardan-Binomial (JB), Janardan-Geometric (JG), Janardan-Poisson (JP) and the Janardan-Logarithmic (JL) distributions, are introduced. In addition, the JP distribution is analyzed in details. Eventually, an example of the proposed class applied on some real data set.

Keywords: Hazard function, Janardan distribution, Moments, Power series, order statistics.

* Corresponding author: Hossein Zamani
Email: zamani.huni@gmail.com

1. Introduction

Modeling lifetime data is considered in the area of survival analysis, in which the lifetime of biological organisms or mechanical systems is investigated. Many recently introduced distributions have the capability to model these types of data appropriately. The underlying logic and assumptions behind these models is that a lifetime of a system with N (discrete random variable) components and the positive continuous random variable, X_i , which designates the lifetime of the i^{th} component, can be described by a non-negative random variable $X = \min(X_1, \dots, X_N)$ or $X = \max(X_1, \dots, X_N)$, depending on whether the components are series or parallel.

Some prominent lifetime distributions are the Exponential Geometric (EG), Exponential Poisson (EP), Exponential Logarithmic (EL), Weibull Geometric (WG) and Weibull Poisson (WP), Lindley Logarithmic (LL) distributions, obtained and proposed by Adamidis and Loukas (1998), Kus (2007), Tahmasbi and Rezaei (2008), Barreto-Souza et al. (2011), Lu and Shi (2011).

The power series class of distributions was examined and derived by Noack (1950), in which N is a discrete random variable depending on the class of power series distributions with probability mass function

$$P(N = n) = \frac{a_n \lambda^n}{C(\lambda)}, n = 1, 2, \dots \quad (1)$$

where $a_n \geq 0$ for all $n = 1, 2, \dots$ which only relies on n , $C(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$ and $\lambda > 0$, is fixed in a way that $C(\lambda)$ is finite and its first derivative with reference to λ are determined and indicated by $C'(\cdot)$.

Many novel models have been recently developed by authors utilizing the power series class of distributions, which some of them have been designed as a combination of some well-established distributions and the power series class of distributions. For instance; Exponential-Power Series (EPS) distributions (Ghahkandi and Ganjali, 2009), Weibull-Power Series (WPS) distributions (Marais and Barreto-Souza, 2011), Complementary Exponential-Power Series (CEPS) distributions (Flores et al., 2011), Generalized Exponential-Power Series (GEPS) distributions (Mahmoudi and Jafari, 2012), Extended Weibull-Power Series (EWPS) distributions (Silva et al., 2013), Generalized Extended Weibull-Power Series (GEWPS) distributions (Alkarni, 2016), Kumaraswamy-Power Series (KPS) distributions (Bidram and Nekoubou, 2013), Linear Failure Rate-Power Series (LFRPS) distributions (Mahmoudi and Jafari, 2014) and Lindley-Power series (LPS) distributions (Warahena-Liyanage and Mavis Pararai, 2015).

Shanker et al. (2013) proposed a combination of exponential $(\frac{\theta}{\alpha})$ and gamma $(2, \frac{\theta}{\alpha})$ distributions which is known as the Janardan distribution (JD). The Janardan distribution

is a continuous distribution with two parameters including $\alpha > 0$ and $\theta > 0$ and is completely characterized by its cumulative distribution function (cdf)

$$G(x; \alpha, \theta) = 1 - \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}. \quad (2)$$

for $x > 0, \alpha > 0$ and $\theta > 0$.

For $\alpha = 1$, the distribution reduces to the one parameter Lindley distribution. It has been proved that the Janardan distribution is a better model compared to the one parameter Lindley distribution, in terms of analyzing waiting time, survival time and grouped mortality data.

The rest of this paper is structured as follows. Section 2 introduces the JPS class of distributions, density, survival, hazard rate and moment generating functions as well as the moments, quantiles and order statistics. In section 3, the special cases of the JPS distributions including Janardan-Binomial (JB) distribution, Janardan-Geometric (JG) distribution, Janardan-Poisson (JP) distribution and Janardan-Logarithmic (JL) distribution are presented. Furthermore, section 4 examines the Janardan-Poisson (JP) distribution in greater detail. Section 5 demonstrates an example of the JP distribution using real data samples. Conclusively, section 6 represents some definitive comments on the subject.

2. The JPS class of distributions

Suppose that X_1, \dots, X_N are independent and identically distributed (iid) random variables following a Janardan distribution with cdf (2) and N , is a member of the power series family with the probability mass function given by (1), independent of X_i . If the random variable X define as $X = \min(X_1, \dots, X_N)$, then the conditional cdf and pdf of $(X | N = n)$ are respectively defined as:

$$G(x; n) = 1 - e^{-\frac{n\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right)^n \quad (2)$$

and

$$g(x; n) = \frac{n\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{n\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right)^{n-1}. \quad (3)$$

for $x > 0$.

Then the Janardan-Power Series (JPS) class of distributions is given by the marginal cdf of X :

$$F(x) = 1 - \frac{C \left[\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) \right]}{C(\lambda)} \quad (4)$$

for $x > 0, \alpha > 0, \theta > 0, \lambda > 0$, in which a random variable X denoted by $X \sim JPS(\alpha, \theta, \lambda)$ follows the JPS distribution.

2.1 Density and survival function

The probability density function of a random variable X following a JPS distribution is given by

$$f(x) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha}x} \frac{C' \left[\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) \right]}{C(\lambda)}; \quad x > 0. \quad (5)$$

The survival function is given by

$$\bar{F}(x) = \frac{C \left[\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) \right]}{C(\lambda)}$$

for $x > 0, \alpha > 0, \theta > 0, \lambda > 0$.

Here some properties of the density (5) are analyzed. The limiting behavior and some other characteristics of the JPS distribution are studied in the following proposition.

Proposition 2.1. The Janardan distribution is a particular limiting case of the JPS distribution when $\lambda \rightarrow 0^+$.

Proof. Using $C(\lambda) = \sum_{n=1}^{\infty} a_n \lambda^n$ in (4), the following function is obtained:

$$\begin{aligned} \lim_{\lambda \rightarrow 0^+} F(x) &= 1 - \lim_{\lambda \rightarrow 0^+} \frac{C \left[\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) \right]}{C(\lambda)}, \\ &= 1 - \lim_{\lambda \rightarrow 0^+} \frac{\sum_{n=1}^{\infty} a_n \lambda^n e^{-\frac{n\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right)^n}{\sum_{n=1}^{\infty} a_n \lambda^n}. \end{aligned}$$

and using *L'Hôpital's rule*, it follows that

$$\begin{aligned} \lim_{\lambda \rightarrow 0^+} F(x) &= 1 - \lim_{\lambda \rightarrow 0^+} \frac{a_1 \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta}{\alpha}x} + \sum_{n=2}^{\infty} n a_n \lambda^{n-1} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right)^n e^{-\frac{n\theta}{\alpha}x}}{a_1 + \sum_{n=2}^{\infty} n a_n \lambda^{n-1}} \\ &= 1 - \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta}{\alpha}x} = G(x; \alpha, \theta). \end{aligned}$$

Proposition 2.2. The density function of JPS class can be represented as an infinite linear combination of the density of minimum order statistic of the i.i.d random variables following the Janardan distribution.

Proof. By using $C'(\lambda) = \sum_{n=1}^{\infty} na_n\lambda^{n-1}$ in (5), it follows

$$f(x) = \sum_{n=1}^{\infty} P(N = n)g(x; n)$$

Where $g(x;n)$ is the pdf of $Y = \min(X_1, \dots, X_n)$ given by (3).

Proposition 2.3. For the pdf of the JPS distribution, we have

$$\lim_{x \rightarrow 0^+} f(x) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} \frac{C'(\lambda)}{C(\lambda)},$$

and

$$\lim_{x \rightarrow \infty} f(x) = 0.$$

2.2 Hazard and reversed hazard functions

The hazard and reversed hazard functions of the JPS distributions are respectively given by

$$h(x; \alpha, \theta, \lambda) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha}x} \frac{C' \left[\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta x}{\alpha}} \right]}{C \left[\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta x}{\alpha}} \right]}, \quad (6)$$

and

$$r(x; \alpha, \theta, \lambda) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha}x} \frac{C' \left[\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta x}{\alpha}} \right]}{C(\lambda) - C \left[\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta x}{\alpha}} \right]}, \quad (7)$$

where $x > 0, \alpha > 0, \theta > 0$ and $\lambda > 0$.

2.3 Quantiles, moments and order statistics

The p^{th} quantile of the JPS distributions, for instance x_p , is given by

$$X_p = \frac{-\alpha}{\theta} - \frac{1}{\alpha} - \frac{\alpha}{\theta} W \left[\frac{-\left(1 + \frac{\theta}{\alpha^2}\right) C^{-1}((1-p)C(\lambda))}{\lambda e^{\frac{\theta}{\alpha^2}+1}} \right]. \quad (8)$$

for $p \in (0,1)$, $C^{-1}(\cdot)$ is the inverse function of $C(\cdot)$ and $W(\cdot)$ is the negative branch of the Lambert W function. More details are available at Corless et al. (1996).

Proposition 2.4. The k^{th} moment of JPS distributions is given by

$$E(X^k) = \frac{\theta^2}{C(\lambda)} \sum_{n=1}^{\infty} a_n \lambda^n \frac{n\alpha^{2n-3}}{(\theta + \alpha^2)^n} L_1(\alpha, \theta, n, k) \quad (9)$$

Where

$$L_1(\alpha, \theta, n, k) = \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \binom{n-1}{i} \binom{i+1}{j} \frac{\alpha^{2j-2i+k+1} \Gamma(j+k+1)}{\theta^{j-i+k+1} n^{j+k+1}}.$$

Proof. We have

$$E(X^k) = \sum_{n=1}^{\infty} P(N = n) E(Y^k).$$

where $Y = \min(X_1, \dots, X_n)$ with the pdf of $g(x; n)$.

or,

$$\begin{aligned} E(X^k) &= \sum_{n=1}^{\infty} P(N = n) \int_0^{\infty} x^k g(x; n) dx, \\ &= \sum_{n=1}^{\infty} P(N = n) \frac{n\alpha^{2n-3}}{(\theta + \alpha^2)^n} \int_0^{\infty} x^k (1 + \alpha x) \left(1 + \frac{\theta}{\alpha^2} (1 + \alpha x)\right)^{n-1} e^{-\frac{n\theta}{\alpha} x} dx, \\ &= \sum_{n=1}^{\infty} P(N = n) \frac{n\alpha^{2n-3}}{(\theta + \alpha^2)^n} L_1(\alpha, \theta, n, k). \end{aligned}$$

where

$$\begin{aligned} L_1(\alpha, \theta, n, k) &= \int_0^{\infty} x^k (1 + \alpha x) \left(1 + \frac{\theta}{\alpha^2} (1 + \alpha x)\right)^{n-1} e^{-\frac{n\theta}{\alpha} x} dx, \\ &= \int_0^{\infty} x^k e^{-\frac{n\theta}{\alpha} x} \left[\sum_{i=0}^{n-1} \binom{n-1}{i} \left(\frac{\theta}{\alpha^2}\right)^i (1 + \alpha x)^{i+1} \right] dx, \\ &= \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \binom{n-1}{i} \binom{i+1}{j} \left(\frac{\theta}{\alpha^2}\right)^i \alpha^j \int_0^{\infty} x^{j+k} e^{-\frac{n\theta}{\alpha} x} dx, \end{aligned}$$

$$= \sum_{i=0}^{n-1} \sum_{j=0}^{i+1} \binom{n-1}{i} \binom{i+1}{j} \frac{\alpha^{2j-2i+k+1} \Gamma(j+k+1)}{\theta^{j-i+k+1} n^{j+k+1}}.$$

2.4 Order statistics

If X_1, \dots, X_n are random variables from a JPS distribution and the k^{th} order statistic is denoted by $X_{k:n}$, $k = 1, 2, \dots, n$, then the pdf of $X_{k:n}$ is given by

$$f_{k:n}(x) = \frac{1}{B(k, n-k+1)} f(x) [F(x)]^{k-1} [1-F(x)]^{n-k}, \quad (8)$$

Where $F(\cdot)$ and $f(\cdot)$ are the cdf and pdf of JPS distributions respectively given by (4) and (5), eq. (8) can be written as follows

$$f_{k:n}(x) = \frac{1}{B(k, n-k+1)} \sum_{i=0}^{n-k} \binom{n-k}{i} (-1)^i f(x) [F(x)]^{i+k-1},$$

In view of the fact that

$$f(x) [F(x)]^{i+k-1} = \frac{1}{i+k} \frac{d}{dx} [F(x)]^{i+k},$$

The corresponding cdf of $f_{k:n}(x)$, denoted by $F_{k:n}(x)$, is given by

$$\begin{aligned} F_{k:n}(x) &= \frac{1}{B(k, n-k+1)} \sum_{i=0}^{n-k} \frac{\binom{n-k}{i} (-1)^i}{i+k} [F(x)]^{i+k}, \\ &= \frac{1}{B(k, n-k+1)} \sum_{i=0}^{n-k} \frac{\binom{n-k}{i} (-1)^i}{i+k} \left[1 - \frac{C \left(\lambda e^{-\frac{\theta}{\alpha} x} \left(1 + \frac{\theta \alpha x}{\theta + \alpha^2} \right) \right)}{C(\lambda)} \right]^{i+k}, \\ &= \frac{1}{B(k, n-k+1)} \sum_{i=0}^{n-k} \frac{\binom{n-k}{i} (-1)^i}{i+k} F_J(x; \alpha, \theta, \lambda, i+k) \end{aligned} \quad (9)$$

Where J is described by an exponential JPS distribution with parameters α, θ, λ and $i+k$.

Thus, the cdf of the k^{th} order statistic can be evaluated as a finite linear combination of the cdf of the exponential JPS distribution.

Expressions for the moment of the k^{th} order statistics $X_{k:n}$, $k = 1, 2, \dots, n$, with cdf (9), can be obtained using a result of Barakat et al. (2004) as follows:

$$\begin{aligned}
E(X_{k:n}^r) &= \sum_{i=n-k+1}^n (-1)^{i-n+k-1} \binom{i-1}{n-k} \binom{n}{i} \int_0^\infty x^{r-1} [\bar{F}(x)]^i dx, \\
&= \sum_{i=n-k+1}^n (-1)^{i-n+k-1} \binom{i-1}{n-k} \binom{n}{i} \int_0^\infty x^{r-1} \left[\frac{C\left(\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right)\right)}{C(\lambda)} \right]^i dx.
\end{aligned} \tag{10}$$

for $r = 1, 2, \dots$ and $k = 1, 2, \dots, n$,

An application of the first moment of order statistics can be utilized in calculating the L-moments, which are in fact the linear combinations of the expected order statistics. For more details see Hosking (1990).

3. Special cases of the JPS class of distributions

Some particular cases of the class of JPS distribution including the Janardan-Binomial (JB), Janardan-Geometric (JG), Janardan-Logarithmic (JL) and Janardan-Poisson (JP) are analyzed in the following section.

In order to obtain the p.d.f, hazard function and moment, the JP distribution is analyzed. To determine the flexibility of the JP distribution, plots of the density and hazard rate functions are presented in Figures 1 and 2 respectively for some selected values of the parameters.

3.1 Janardan-Binomial distribution

The Janardan-binomial (JB) distribution is determined by the cdf (4) with $C(\lambda) = (1 + \lambda)^m - 1$, and is given by

$$F(x) = 1 - \frac{\left[\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) + 1 \right]^m - 1}{(1 + \lambda)^m - 1}. \tag{11}$$

Where $x > 0$, $0 < \lambda < 1$ and m is positive integer.

The corresponding p.d.f and hazard rate function are respectively given by

$$f(x) = \frac{m\lambda\theta^2}{\alpha(\theta + \alpha^2)} e^{-\frac{\theta}{\alpha}x} (1 + \alpha x) \frac{\left[\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) + 1 \right]^{m-1}}{(1 + \lambda)^m - 1}. \tag{12}$$

and

$$h(x) = \frac{m\lambda\theta^2}{\alpha(\theta + \alpha^2)} e^{-\frac{\theta}{\alpha}x} (1 + \alpha x) \frac{\left[\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) + 1 \right]^{m-1}}{\left[\lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) + 1 \right]^m - 1}. \quad (13)$$

Where $x > 0, \alpha > 0, \theta > 0, \lambda > 0$ and m is positive integer.

3.2 Janardan -Geometric distribution

The Janardan-geometric (JG) distribution is characterized by the cdf (4) with $C(\lambda) = \lambda(1 - \lambda)^{-1}$, which is given by

$$F(x) = 1 - \frac{e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) \left[1 - \lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) \right]^{-1}}{(1 - \lambda)^{-1}}. \quad (14)$$

$x > 0, \alpha > 0, \theta > 0, 0 < \lambda < 1.$

The corresponding pdf and hazard rate function are respectively given by

$$f(x) = \frac{\theta^2}{\alpha(\theta + \alpha^2)} e^{-\frac{\theta}{\alpha}x} (1 + \alpha x) \frac{(1 - \lambda)}{\left[1 - \lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) \right]^2}. \quad (15)$$

and

$$h(x) = \frac{\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) \frac{\left[1 - \lambda e^{-\frac{\theta}{\alpha}x} \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) \right]^{-1}}{\left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right)}. \quad (16)$$

where $x > 0, \alpha > 0, \theta > 0, 0 < \lambda < 1.$

3.3 Janardan-Logarithmic distribution

The Janardan-Logarithmic (JL) distribution is characterized by the cdf (4) with $C(\lambda) = -\log(1 - \lambda)$, which given by

$$F(x) = 1 - \frac{\log \left[1 - \lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta}{\alpha}x} \right]}{\log(1 - \lambda)}. \quad (17)$$

for $x > 0, \alpha > 0, \theta > 0, 0 < \lambda < 1.$

The associated p.d.f and hazard rate function are respectively given by

$$f(x) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha}x} \frac{\left[1 - \lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta}{\alpha}x} \right]^{-1}}{-\log(1 - \lambda)}. \quad (18)$$

and

$$h(x) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha}x} \frac{\left[1 - \lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}\right]^{-1}}{-\log \left[1 - \lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}\right]}, \quad (19)$$

for $x > 0, \alpha > 0, \theta > 0, 0 < \lambda < 1$.

4. Janardan-Poisson distribution

This section presents a particular case of JPS class of distributions, the Janardan-poisson (JP) distribution which will be later explained in detail. The properties of this distribution are originated from the properties of the general class.

4.1 Survival, density, hazard and reverse hazard functions

The Janardan-Poisson (JP) distribution is characterized by the cdf (4) with $C(\lambda) = e^\lambda - 1$ and $a_n = \frac{1}{n!}$, where $\lambda > 0$. The following equation gives us the survival function of JP distribution:

$$\bar{F}(x) = \frac{\exp\left\{\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}\right\} - 1}{e^\lambda - 1}, \quad (20)$$

for $x > 0, \alpha > 0, \theta > 0, \lambda > 0$.

The corresponding p.d.f, hazard and reverse functions are respectively given by

$$f(x) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha}x} \frac{\exp\left\{\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}\right\}}{e^\lambda - 1}, \quad (21)$$

and

$$h(x) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha}x} \frac{\exp\left\{\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}\right\}}{\exp\left\{\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}\right\} - 1}, \quad (22)$$

and

$$r(x) = \frac{\lambda\theta^2}{\alpha(\theta + \alpha^2)} (1 + \alpha x) e^{-\frac{\theta}{\alpha}x} \frac{\exp\left\{\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}\right\}}{e^\lambda - \exp\left\{\lambda \left(1 + \frac{\theta\alpha x}{\theta + \alpha^2}\right) e^{-\frac{\theta}{\alpha}x}\right\}}, \quad (23)$$

where $x > 0, \alpha > 0, \theta > 0, \lambda > 0$.

Figures 1 and 2 illustrates the plots for the p.d.f and hazard function of the JP distribution respectively for a number of combinations of the parameters α, θ, λ . The plots indicate that the JP distribution can be decreasing or right skewed. The JP distribution has a positive asymmetry. Hence it can be a candidate for modeling positive skewed data.

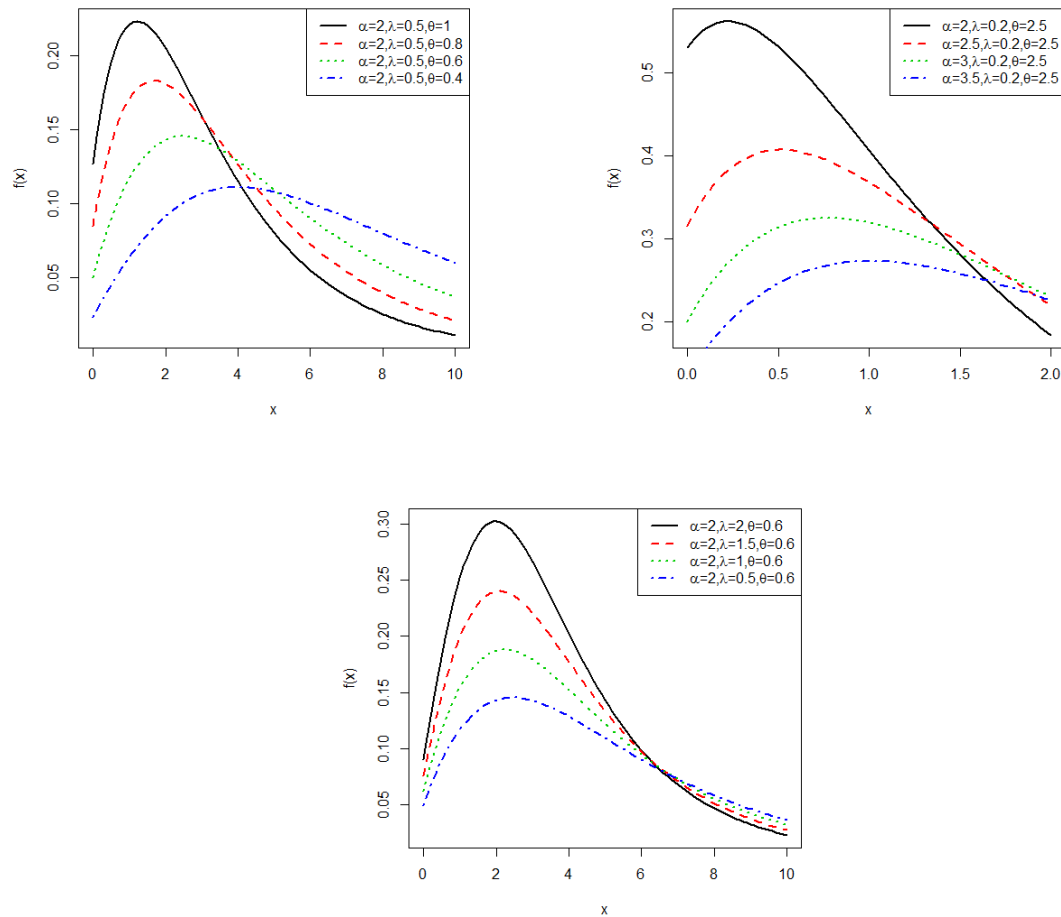


Figure 1: Plots of the pdf for different values of α, θ, λ .

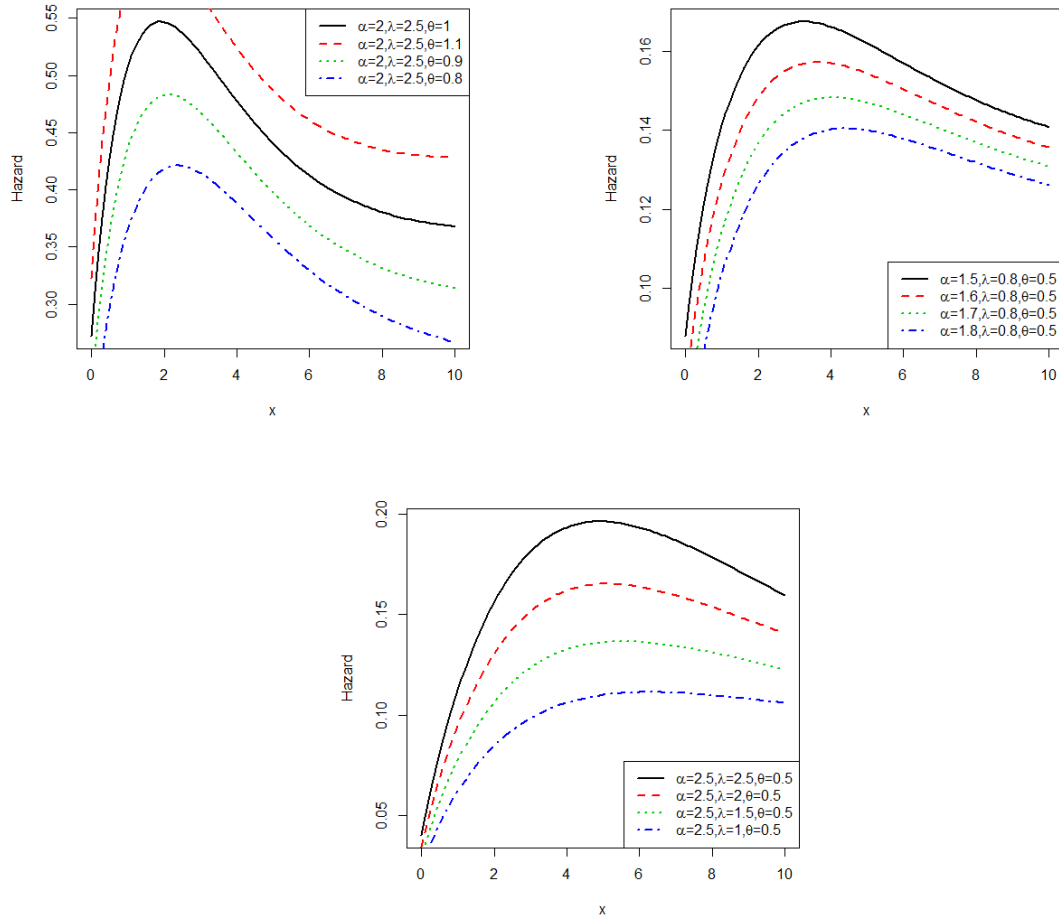


Figure 2: Plots of the hazard rate function for different value of α, θ, λ .

The plots for the hazard function of JP distribution exhibit different shapes including monotonically increasing, increasing-decreasing and increasing-decreasing-increasing shapes. These interesting shapes of the hazard function indicate that JP distribution is suitable for monotonic and non-monotonic hazard behaviors which are more likely to be encountered in real life situations.

4.2 Quantiles and median

By substituting $C^{-1}(\lambda) = \log(1 + \lambda)$ in equation (6), the quantiles and median for the JP distribution are respectively given as

$$X_p = \frac{-\alpha}{\theta} - \frac{1}{\alpha} - \frac{\alpha}{\theta} W \left[\frac{-\left(1 + \frac{\theta}{\alpha^2}\right) \log\left(1 + (1-p)(e^\lambda - 1)\right)}{\lambda e^{\frac{\theta}{\alpha^2} + 1}} \right], \quad 0 < p < 1, \quad (24)$$

and

$$m = \frac{-\alpha}{\theta} - \frac{1}{\alpha} - \frac{\alpha}{\theta} W \left[\frac{-\left(1 + \frac{\theta}{\alpha^2}\right) \log \left(1 + \frac{1}{2}(e^\lambda - 1)\right)}{\lambda e^{\frac{\theta}{\alpha^2} + 1}} \right], \quad (25)$$

with $W(\cdot)$ as the negative branch of the Lambert W function.

4.3 Moments and moment generating function

The k^{th} moment of a random variable X from the JP distribution is given by

$$E(X^k) = \sum_{n=1}^{\infty} \frac{\lambda^n \theta^2 \alpha^{2n-3} L_1(\alpha, \theta, n, k)}{(n-1)! (e^\lambda - 1) (\theta + \alpha^2)^n} \quad (26)$$

The first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) of the JP distribution for some selected values of the parameters α, θ, λ are provided in table 1.

Table 1: Moments of JP distribution for some values of α, θ, λ .

μ'_k	$\hat{\beta} = 1.5, \hat{\lambda} = 0.2,$ $\hat{\alpha} = 1$	$\hat{\beta} = 1.5, \hat{\lambda} = 0.2,$ $\hat{\alpha} = 1.5$	$\hat{\beta} = 2.5, \hat{\lambda} = 0.8,$ $\hat{\alpha} = 1.5$	$\hat{\beta} = 2.5, \hat{\lambda} = 0.8,$ $\hat{\alpha} = 2.5$
μ'_1	0.8897	1.5296	0.7283	1.4347
μ'_2	1.4876	4.1027	1.0380	3.6542
μ'_3	3.5880	15.4453	2.1921	13.3277
μ'_4	11.2268	74.4287	6.0953	63.1371
SD	0.8342	1.3277	0.7124	1.2632
CV	0.9376	0.8680	0.9781	0.8804
CS	1.7672	1.6134	1.9269	1.7393
CK	7.5231	6.7901	8.4194	7.4902

The moment generating function (mgf) of the JP distribution is given by

$$\begin{aligned} M(t) &= \sum_{k=0}^{\infty} \frac{t^k}{k!} E(X^k) \\ &= \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\lambda^n \theta^2 \alpha^{2n-3} L_1(\alpha, \theta, n, k)}{(n-1)! (e^\lambda - 1) (\theta + \alpha^2)^n}. \end{aligned} \quad (27)$$

4.4 Cdf and p.d.f of order statistics

By using the cdf of the JP distribution as an alternative to (9), the cdf and p.d.f of the k^{th} order statistics respectively provided as

$$F_{k:n}(x) = \frac{1}{B(k, n-k+1)} \frac{\sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i}}{i+k} \left(1 - \frac{\exp \left\{ \lambda \left(1 + \frac{\theta \alpha x}{\theta + \alpha^2} \right) e^{-\frac{\theta}{\alpha} x} \right\} - 1}{e^\lambda - 1} \right)^{i+k}, \quad (28)$$

and

$$f_{k:n}(x) = \frac{1}{B(k, n-k+1)} \frac{\sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i}}{i+k} \frac{\lambda \theta^2}{\alpha(\theta+\alpha^2)} (1+\alpha x) e^{-\frac{\theta}{\alpha}x} \frac{\exp\left\{\lambda\left(1+\frac{\theta\alpha x}{\theta+\alpha^2}\right)e^{-\frac{\theta}{\alpha}x}\right\}}{(e^{\lambda}-1)^{i+k}} \\ \times \left(e^{\lambda} - \exp\left\{\lambda\left(1+\frac{\theta\alpha x}{\theta+\alpha^2}\right)e^{-\frac{\theta}{\alpha}x}\right\} \right)^{i+k-1}, \quad (29)$$

Where $x > 0, \alpha > 0, \theta > 0, \lambda > 0$.

5. Maximum Likelihood Estimation

Let x_1, \dots, x_n be the observations of a random sample of size n from the $JPS(\alpha, \theta, \lambda)$ distributions. The log-likelihood function for the vector of $\Theta = (\alpha, \theta, \lambda)^T$ is given by

$$\ell_n = \ell_n(\Theta, x_1, \dots, x_n) = n \log(\lambda) + 2n \log(\theta) - n \log(\alpha) \\ - n \log(\theta + \alpha^2) + \sum_{i=1}^n \ln C'(\lambda p_i) - n \ln C(\lambda) \\ + \sum_{i=1}^n \ln(1 + \alpha x_i) - \frac{\theta}{\alpha} \sum_{i=1}^n x_i \quad (30)$$

where $p_i = e^{-\frac{\theta}{\alpha}x_i} \left(1 + \frac{\theta\alpha x_i}{\theta+\alpha^2}\right)$.

The score vector is given by $U_n(\Theta) = \left(\frac{\partial \ell_n}{\partial \alpha}, \frac{\partial \ell_n}{\partial \theta}, \frac{\partial \ell_n}{\partial \lambda}\right)^T$ with the components

$$\frac{\partial \ell_n}{\partial \alpha} = \frac{-n}{\alpha} - \frac{2n\alpha}{\theta + \alpha^2} + \sum_{i=1}^n \frac{\left(\lambda p_i \left(\frac{\theta x_i}{\alpha^2} + \frac{\theta x_i(\theta - \alpha^2)}{(\theta + \alpha^2 + \theta\alpha x_i)(\theta + \alpha^2)}\right)\right) C''(\lambda p_i)}{C'(\lambda p_i)} \\ + \sum_{i=1}^n \frac{x_i}{1 + \alpha x_i} + \frac{\theta}{\alpha^2} \sum_{i=1}^n x_i, \\ \frac{\partial \ell_n}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{\theta + \alpha^2} + \sum_{i=1}^n \frac{\left(\lambda p_i \left(\frac{\alpha^3 x_i}{(\theta + \alpha^2 + \theta\alpha x_i)(\theta + \alpha^2)} - \frac{x_i}{\alpha}\right)\right) C''(\lambda p_i)}{C'(\lambda p_i)} - \frac{1}{\alpha} \sum_{i=1}^n x_i, \\ \frac{\partial \ell_n}{\partial \lambda} = \frac{n}{\lambda} + \sum_{i=1}^n \frac{p_i C''(\lambda p_i)}{C'(\lambda p_i)} - \frac{n C'(\lambda)}{C(\lambda)}.$$

The maximum likelihood estimation of $\Theta = (\alpha, \theta, \lambda)^T$ is obtained by solving the nonlinear System of $U_n(\hat{\Theta}) = 0$. The solutions can be obtain by using a numerical method (such as Quasi Newton algorithm).

For instance, we have computed MLEs for JP distribution and some values of parameters for different sample sizes $n = 5, 10, 20, 30, 50, 100$. First of all, we generated values of JP distribution by using of solving equation $F(x) = u$ where $F(\cdot)$ is cumulative distribution function of JP distribution and u is a value of uniform distribution. Unfortunately, this equation does not have a solution in closed form and therefore it is solved by numerical methods. Here, we use function uniroot in R software for this end. Then, we minimize $-\log(\ell_n)$ numerically in R by using of function optim and method “L-BFGS-B”.

In the following tables, some results have shown for some parameters.

Table 2. Results of MLE for small value of parameters.

n	5	10	20	30	50	100
θ	0.1	0.1	0.1	0.1	0.1	0.1
α	0.2	0.2	0.2	0.2	0.2	0.2
λ	0.5	0.5	0.5	0.5	0.5	0.5
$\hat{\theta}$	0.908	0.700	0.542	0.437	0.342	0.245
$\hat{\alpha}$	1.109	0.930	0.790	0.677	0.564	0.439
$\hat{\lambda}$	0.438	0.574	0.656	0.697	0.694	0.697
$MSE(\hat{\theta})$	1.227	0.850	0.556	0.387	0.238	0.107
$MSE(\hat{\alpha})$	1.518	1.223	0.962	0.754	0.527	0.295
$MSE(\hat{\lambda})$	0.454	0.517	0.543	0.524	0.477	0.418

Table 3. Results of MLE for different value of parameters.

n	5	10	20	30	50	100
θ	1	1	1	1	1	1
α	5	5	5	5	5	5
λ	2	2	2	2	2	2
$\hat{\theta}$	1.279	1.236	1.188	1.175	1.146	0.818
$\hat{\alpha}$	6.722	6.136	5.703	5.615	5.444	3.838
$\hat{\lambda}$	2.293	1.954	1.831	1.831	1.893	1.828
$MSE(\hat{\theta})$	0.622	0.651	0.634	0.611	0.582	0.228
$MSE(\hat{\alpha})$	18.180	16.707	14.996	14.305	13.089	4.557
$MSE(\hat{\lambda})$	4.967	4.206	3.625	3.338	2.981	1.882

Table 4. Results of MLE for same value of parameters.

n	5	10	20	30	50	100
θ	1	1	1	1	1	1
α	1	1	1	1	1	1
λ	1	1	1	1	1	1
$\hat{\theta}$	1.098	1.019	0.971	0.951	0.946	0.909
$\hat{\alpha}$	0.892	0.860	0.872	0.876	0.885	0.884
$\hat{\lambda}$	0.447	0.582	0.720	0.763	0.780	0.847
MSE($\hat{\theta}$)	0.366	0.375	0.361	0.349	0.319	0.286
MSE($\hat{\alpha}$)	0.354	0.324	0.320	0.310	0.283	0.264
MSE($\hat{\lambda}$)	0.825	0.790	0.767	0.735	0.705	0.640

Table 5. Results of MLE for large value of parameters.

n	5	10	20	30	50	100
θ	10	10	10	10	10	10
α	7	7	7	7	7	7
λ	5	5	5	5	5	5
$\hat{\theta}$	7.934	7.900	8.210	8.466	8.672	9.051
$\hat{\alpha}$	6.059	5.896	5.960	5.998	6.136	6.243
$\hat{\lambda}$	5.942	5.563	5.403	5.302	5.397	5.339
MSE($\hat{\theta}$)	24.129	23.379	20.869	19.096	17.169	16.012
MSE($\hat{\alpha}$)	10.869	10.886	9.628	8.606	7.035	5.524
MSE($\hat{\lambda}$)	9.460	9.036	8.078	7.716	6.696	5.370

As all tables show, from values of MSE, estimation for large sample size (n) is better than small sample size. Also, small values of parameters have more reliable estimation than large values of parameters. Finally, we state that MLEs are not good for some values of parameters and sample sizes. These have two reasons in statistically point of view. Two reasons are both related to numerical computations in steps data generation and minimizing. In future works we are going to seek a Bayesian estimation method whose precise is better than MLE method.

6. Applications

In order to determine the flexibility of Janardan-Poisson as a possible alternative to Lindley-Poisson as well as Lindley distributions, we provided two examples in the

following section. For both examples, the data histogram, probabilities and the plots of fitted densities are presented with real data.

Example 1:

The waiting times (in minutes) to receive banking services for 100 customers are provided in the first data set by Ghitany et al.(2008). Estimates of parameters of JP, LP and Lindley distributions with AIC are provided in Table 6.

Table 6: Estimates of models for waiting data.

Distributions	β	λ	α	$-2\log L$
Lindley	0.187	-	-	643.39
Lindley-Poisson	0.057	7.367	-	642.3
Janardan-Poisson	3.717	0.526	20.3	639.79

The AIC of the fitted models indicates that the JP leads to a better fitting to data compared to LP and Lindley distributions. The plots of fitted densities and histogram of data are given in Figure3.

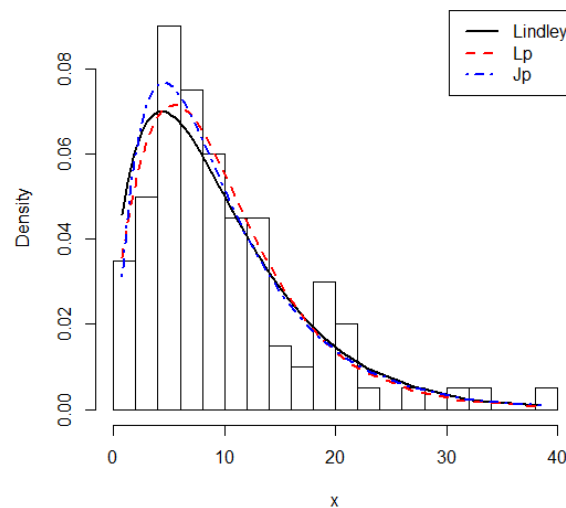


Figure 3: Histogram and fitted density plots for waiting data.

Example 2:

This data set represents the lifetime's data relating to relief times (in minutes) of 20 patients receiving an analgesic which reported by Gross et al. (1975) and applied by Shankar et.al (2016).

Figure 4 illustrates the fitted densities plot as well as the data histogram.

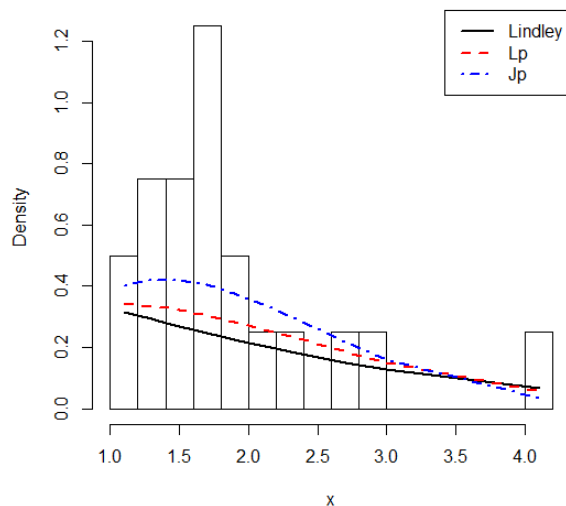


Figure 4: Histogram and fitted density plots for life time data.

Table 7 represents the results of the AIC along with the parameters of JP, LP and Lindley distributions.

Table 7: Estimates of models for life time data.

Distributions	β	λ	α	$-2\log L$
Lindley	0.8161	-	-	62.48
Lindley-Poisson	0.0192	709.76	-	57.6
Janardan-Poisson	4.125	694.7	152.66	51.32

The AIC of the fitted models revealed that the JP is markedly better fitted to the dataset in comparison to LP and Lindley distributions.

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