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Exponentiated Weibull-Lomax Distribution: Properties and Estimation

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Abstract: In this article, we introduce a new class of five-parameter model called the Exponentiated Weibull Lomax arising from the Exponentiated Weibull generated family. The new class contains some existing distributions as well as some new models. Explicit expressions for its moments, distribution and density functions, moments of residual life function are derived. Furthermore, Rényi and q–entropies, probability weighted moments, and order statistics are obtained. Three suggested procedures of estimation, namely, the maximum likelihood, least squares and weigthed least squares are used to obtain the point estimators of the model parameters. Simulation study is performed to compare the performance of different estimates in terms of their relative biases and standard errors. In addition, an application to two real data sets demonstrate the usefulness of the new model comparing with some new models.

Keywords: Exponentiated Weibull-G family of distributions; Lomax distribution, Least sqaures method; Weighted least squares method; Maximum likelihood mehod.

1. Introduction

The Lomax or Pareto II distribution is originally used for modeling business failure data, and it has been widely applied in a variety of contexts studies. Atkinson and Harrison (1978) and Harris (1968) applied the Lomax distribution to income and wealth data. Bryson (1974) suggested Lomax distribution as an alternative to the exponential distribution for heavy-tailed data sets. Myhre and Saunders (1982) applied Lomax distribution in the right censored data. Different procedures of estimation for the Lomax distribution are suggested by Lingappaiah (1986). Moments of record values based on Lomax distribution were discussed by Ahsanullah (1991) and Balakrishnan and Ahsanullah (1994). Order statistics from non-identical right-truncated Lomax distribution and its applications were discussed by Childs et al. (2001). Abd-Elfattah et al. (2007) discussed the Bayesian and non-Bayesian estimation problem of the sample size in the case of type-I censored samples. Based on cumulative exposure model, the optimal times plans of changing stress level of simple stress for the Lomax distribution were determined by Hassan and Al-Ghamdi (2009). Abd-Elfattah and Alharbey (2010) discussed the estimation problem for the Lomax distribution based on generalized probability weighted moments. Nasiri and Hosseini (2012) obtained the Bayesian and non-Bayesian estimators for the Lomax parameters in presence of record values. The estimation problem of the unknown parameters for the Lomax distribution based on type-II progressively hybrid censored samples has been discussed by Ma and Shi (2013). Ahmad et al. (2015) obtained the Bayesian estimators of the shape parameter of the Lomax distribution under different loss functions. The optimal times of changing stress level for k-level step stress accelerated life tests based on adaptive type-II progressive hybrid censoring with product's life time following Lomax distribution have been investigated by Hassan et al. (2016).

The cumulative distribution function (cdf) and the probability density function (pdf) of Lomax distribution are given, respectively, by

$$G\left(x\ ;\ \lambda,\theta\right)=1-(1+\frac{x}{\lambda})^{-\theta},\ x,\lambda,\theta>0 \quad (1)$$

$$g(\left(x\ ;\ \lambda,\theta\right)=\frac{\theta}{\lambda}(1+\frac{x}{\lambda})^{-(\theta+1)},\ x,\lambda,\theta>0 \quad (2)$$

where, $\lambda>0$ is the scale parameter and $\theta>0$ is the shape parameter.In the literature, some of the extended and generalized forms of the Lomax distribution were derived and discussed by several authors. Ghitany et al. (2007) suggested Marshall-Olkin extended Lomax distribution. Abdul-Moniem and Abdel-Hameed (2012) introduced the Exponentiated Lomax distribution by adding shape parameter to the distribution function of Lomax distribution. Elbatal and Kareem (2014) proposed the Kumaraswamy Exponentiated Lomax distribution. Lemonte and Cordeiro (2013) investigated beta Lomax, Kumaraswamy Lomax and McDonald Lomax. Cordeiro et al. (2015) introduced the gamma-Lomax based on gamma generated family. Ashour and Eltehiwy (2013) introduced the transmuted Exponentiated Lomax distribution. Shams (2013) introduced Kumaraswamy-generalized Lomax distribution. Tahir et al. (2016a) introduced the Weibull Lomax based on Weibull generated family. The Gumbel-Lomax has been introduced by Tahir et al. (2016b). Rady et al. (2016) introduced more flexible model through applying power transformation, named as the power Lomax distribution.

In recent years, new generated families of continuous distributions have attracted several statisticians to develop new models. These families are obtained by introducing one or more additional shape parameter(s) to the baseline distribution. A recent family of univariate distributions generated by Exponentiated Weibull random variables was suggested by Hassan and Elgarhy (2016) and then by Cordeiro et al. (2017). The cumulative distribution function of Exponentiated Weibull-generated (EW-G)family is defined by

$$F(x)=[1-exp(-\alpha[\frac{G(x)}{1-G(x)}]^{\beta})]^{a}\ ;\ x>0\ ;\ a,\alpha,\beta>0 \quad (3)$$

where $a,\beta>0$ are the two shape parameters and $\alpha>0$ is the scale parameter. The associated pdf is given by

$$f(x)=\frac{a\alpha\beta(G(x))^{\beta-1}g(x)}{(1-(G(x))^{\beta+1}}e^{-\alpha[\frac{G(x)}{1-G(x)}]^{\beta}}[1-exp(-\alpha\left[\frac{G(x)}{1-G(x)}\right]^{\beta}\Big)]^{a-1};x>0\ ;a,\alpha,\beta>0 \quad (4)$$

Note that;

1. For $\beta=1$, the cdf (3) reduces to the odd generalized exponential family (Tahir et al. (2015).

2. For $\beta=1, a=1$, the cdf (3) reduces to odd exponential-G family (Bourguignon et al. 2014).

3. For $\beta=2,\ \alpha=1$, the cdf (3) reduces to Burr X -G family (Yousof et al. 2017).

In this study, we introduce a new five-parameter model, called the Exponentiated Weibull Lomax distribution based on the *EW-G* family.The rest of the paper is outlined as follows. In Section 2, we introduce the Exponentiated Weibull Lomax (*EWL*) distribution. In Section 3, we derive a very useful representation for the pdf and cdf of the proposed distribution besides some of its mathematical properties. In Section 4, three different methods of point estimation; namely maximum likelihood, least squares and weighted least squares are performed to obtain the point estimates of the model parameters. An extensive simulation study is performed to compare the performance of the different estimators in Section 5. Section 6 provides real data examples to illustrate the applicacbility of *EWL* distribution and finally we conclude the paper in Section 7.

2. Exponentiated Weibull-Lomax Distribution

The cdf of Exponentiated Weibull Lomax distribution, denoted by *EWL* $(a,\alpha,\beta,\lambda,\theta)$, is obtained by inserting cdf (1) in (3) as follows

$$F(x\ ;\Psi)=[1-exp(-\alpha(1+\frac{x}{\lambda})^{\theta}-1)^{\beta}]^{\alpha};a,\alpha,\beta,\lambda,\theta>0,x>0\ (5)$$

where, $\Psi\equiv(a,\alpha,\beta,\lambda,\theta)$, is the set of parameters. The pdf of *EWL* distribution is obtained by inserting the pdf (1) and cdf (2) into (4) as the following

$$f(x\,;\Psi)=\frac{a\alpha\beta\theta}{\lambda}(1+\frac{x}{\lambda})^{\beta\theta-1}(1-(1+\frac{x}{\lambda})^{-\theta})^{\beta-1}exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta}[1-exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta}]^{\alpha-1}(6)$$

A random variable *X* has density (6) will be denoted by $X\sim EWL\,(a,\alpha,\beta,\lambda,\theta)$.

Some special sub-models arise from cdf (5) as follows:

1. For $a=1$, the cdf (5) reduces to Weibull Lomax (WL) (Tahir et al. (2016 a)).

2. For $\beta=1$, the cdf (5) reduces to new model, called odd generalized exponential Lomax.

3. For $\beta=1, a=1$ the cdf (5) reduces to new model, called odd exponential Lomax.

4. For $\beta=2,\ \alpha=1$, the cdf (5) reduces to new model, called Burr X Lomax (BXL)

5. For $Y=\left[(1+\frac{x}{\lambda})^{\theta}-1\right]$, $X\sim EWL\,(a,\alpha,\beta,\lambda,\theta)$, then *Y* has the Exponentiated Weibull distribution (Mudholkar and Srivastava (1993)).

6. For $Y=\left[(1+\frac{x}{\lambda})^{\theta}-1\right]^{\beta}$, $X\sim EWL\,(a,\alpha,\beta,\lambda,\theta)$, then *Y* has the Exponentiated exponential distribution (Gupta and Kundu(1999)).

7. For $Y = \left[(1+\frac{x}{\lambda})^{\theta}-1\right]^{\beta}$, $X \sim EWL\ (1,\alpha,\beta,\lambda,\theta)$, then Y has the exponential distribution.

Figure (1) provides plots of the pdf for some selected values of parameters. It is clear from Figure 1 that the *EWL* densities take various shapes such as symmetrical, right-skewed, reversed J-shaped and unimodal

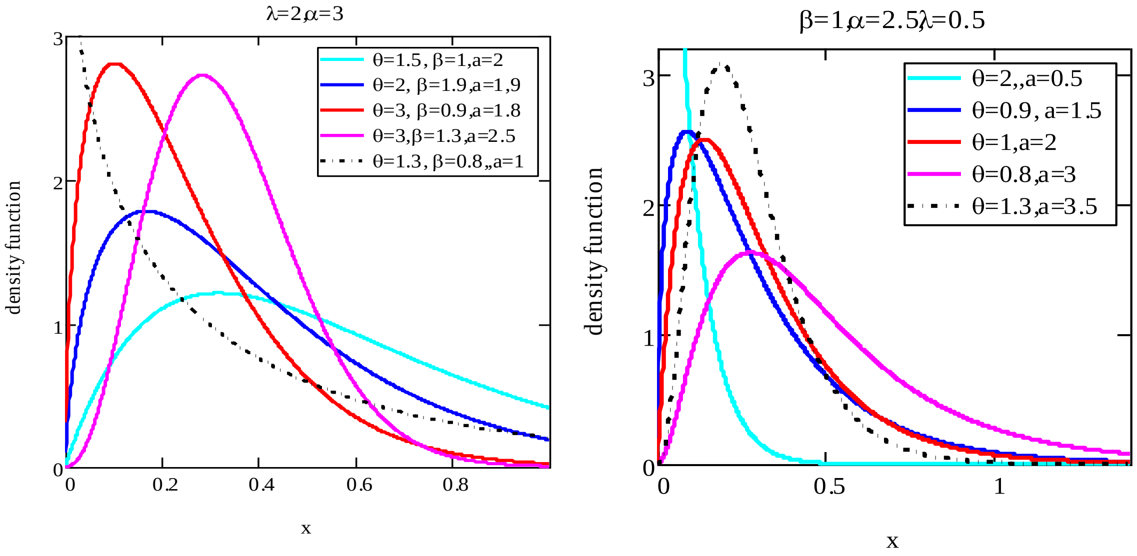


Figure 1: Plots of the *EWL* pdf for some parameters

The survival and hazard rate (hrf) of *EWL* distribution are given, respectively, as follows:

$$R((x\ ;\Psi) = 1-[1-exp(-\alpha(1+\frac{x}{\lambda})^{\theta}-1)^{\beta})]^{\alpha},$$

$$h((x\ ;\Psi) = \frac{a\alpha\beta\theta(1+\frac{x}{\lambda})^{\beta\theta-1}(1-(1+\frac{x}{\lambda})^{-\theta})^{\beta-1}exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta})[1-exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta}]^{\alpha-1}}{\lambda\{1-[1-exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta}]^{\alpha}\}}$$

Furthermore, the reversed-hazard rate and cumulative hazard rate functions are as follows: $\tau(x\ ;\Psi) = \frac{\alpha\beta\theta(1+\frac{x}{\lambda})^{\beta\theta-1}(1-(1+\frac{x}{\lambda})^{-\theta})^{\beta-1}exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta})}{\lambda\{1-[1-exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta}]\}}$

And

$$H(x\ ;\Psi) = -In|\ R((x\ ;\Psi)| = -In|1-[1-exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta}]^{\alpha}|$$

Also, Figure 2 shows that hazard rate shapes can take different shapes such as constant, increasing, decreasing, and reversed J shape. This fact implies that the *EWL* can be very useful for fitting data sets with various shapes.

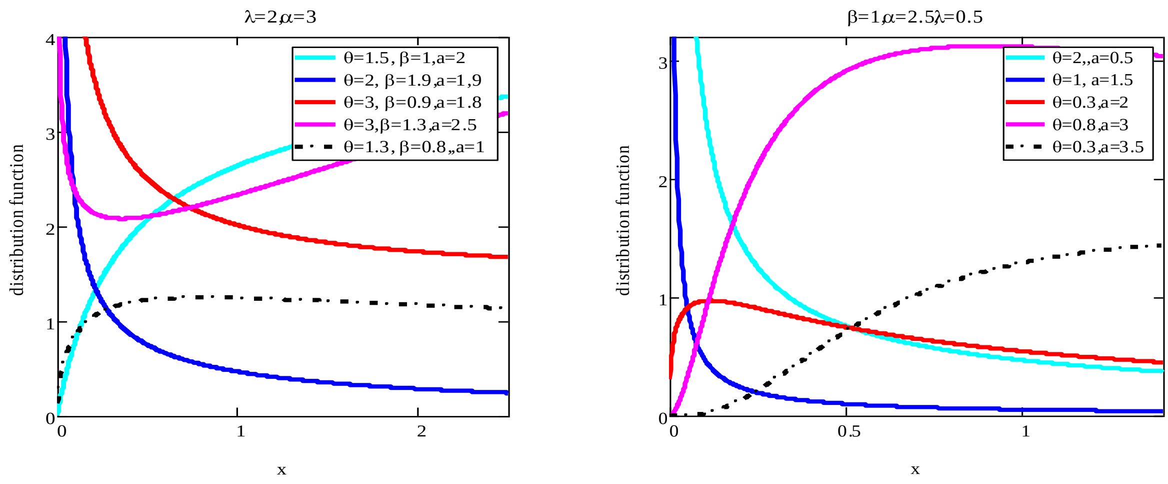


Figure 2: Plots of the *EWL* hrf for some parameters

3. Statistical Properties

This section provides some properties of the *EWL* distribution.

3.1 Useful Expansions

Here, expansions for the pdf and cdf of Exponentiated Weibull-Lomax distribution are derived. The pdf (6) can be rewritten as follows:

$$f(x\,;\Psi)=\frac{\alpha\beta\theta(1-(1+\frac{x}{\lambda})^{-\theta})^{\beta-1}(1+\frac{x}{\lambda})^{-\theta})^{-(\theta+1)}}{\lambda\{1-[1-(1+\frac{x}{\lambda})^{-\theta}]\}^{\beta+1}}\,e^{-\alpha\left(\frac{1-(1+\frac{x}{\lambda})^{-\theta}}{(1+\frac{x}{\lambda})^{-\theta}}\right)^{\beta}}\Big[1-exp\Big(-\alpha\Big(\frac{1-(1+\frac{x}{\lambda})^{-\theta}}{1-[1-(1+\frac{x}{\lambda})^{-\theta}]}\Big)^{\beta}\Big)\Big]^{\alpha-1} \quad (7)$$

Since the generalized binomial theorem, for $\beta > 0$ is real non integer and $|z| < 1$, is given by:

$$(1-z)^{\beta-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{\beta-1}{i}z^{i} \quad (8)$$

Then, by applying the binomial theorem (8),where a is real non integer and the power series for the exponential function in (7), then the pdf of *EWL* distribution becomes:

$$f(x\,;\,\Psi)=\sum_{i,j=0}^{\infty}\binom{a-1}{i}(-1)^{i+j}\frac{a\theta\beta\alpha^{j+1}(i+1)^{j}}{\lambda j!}(1+\frac{x}{\lambda})^{-(\theta+1)}(1-(1+\frac{x}{\lambda})^{-\theta})^{\beta+\beta j-1}\Big[1-\Big(1+\frac{x}{\lambda})^{-\theta}\Big]\Big]^{-\beta(j+1)-1} \quad (9)$$

Also, it is known that

$$(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{\Gamma(k+j)z^{j}}{\Gamma(k)j!},\ |z|<1, k>0 \quad (10)$$

Hence, the pdf of *EWL* distribution takes the following form

$$f(x\,;\Psi)=\sum_{i,j,k=0}^{\infty}w_{i,j,k}h_{\beta(j+1)+k}(x),$$

$$\left\{\begin{aligned} f(x\,;\Psi) &= \sum_{i,j,k=0}^{\infty} w_{i,j,k} h_{\beta(j+1)+k}(x), \\ w_{i,j,k} &= (-1)^{i+j} \frac{a\beta\alpha^{j+1}(i+1)^{j}\Gamma(\beta(j+1)+k)}{j!k!\Gamma(\beta(j+1)+1)}, h_d(x) = dg(x\,;\Psi)(G(x\,;\Psi))^{d-1} \end{aligned}\right\} \quad (11)$$

Therefore, the pdf of Exponentiated Weibull Lomax can be expressed as a mixture of Exponentiated Lomax densities with parameters θ, λ and $\beta(j+1)+k$.

Furtheremore, an expansion for the cumulative distribution function $\left(F(x;\Psi)\right)^{s}$ is derived. Using binomial expansion for $\left(F(x;\Psi)\right)^{s}$, where s is an integer and a is a real non integer , leads to :

$$(F(x\,;\Psi))^{S} = \sum_{p=0}^{\infty} (-1)^{p} \binom{as}{p} exp(-\alpha p \left(\frac{1-(1+\frac{x}{\lambda})^{-\theta}}{1-[1-(1++\frac{x}{\lambda})^{-\theta}]})^{\beta} \right) \quad (12)$$

Applying the power series for the exponential function and the binomial expansion (10) in (12), then we obtain

$$\left\{\begin{aligned} (F(x\,;\Psi))^{S} &= \sum_{p,q,l=0}^{\infty} \eta_{p,q,l} G_{\beta q+1}(x), \\ \eta_{p,q,l} &= (-1)^{p+q} \binom{as}{p} \frac{G_{\beta q+1}(x)\Gamma(\beta q+1)(\alpha p)^{q}}{q!1!\Gamma(\beta q)} \end{aligned}\right\} \quad (13)$$

where $G_{\beta q+\ell}(x)$ is the cdf of the Exponentiated Lomax distribution with parameters λ, θ and $\beta q + \ell$.

3.2 Quantile Function

The quantile function, say $x = Q(u) = F^{-1}(u)$ of X can be obtained by inverting (5) as follows

$$x = Q(u) = \lambda\{[1 + \left[-\frac{1}{\alpha} ln\left(1-u^{\frac{1}{a}}\right)\right]^{\frac{1}{\beta}}]^{\frac{1}{\theta}} - 1\Big\}, \quad (14)$$

where u is a uniform random variable on the unit interval $(0,1)$. In particular, the median is obtained by subsituting $u = 0.5$ in (14).

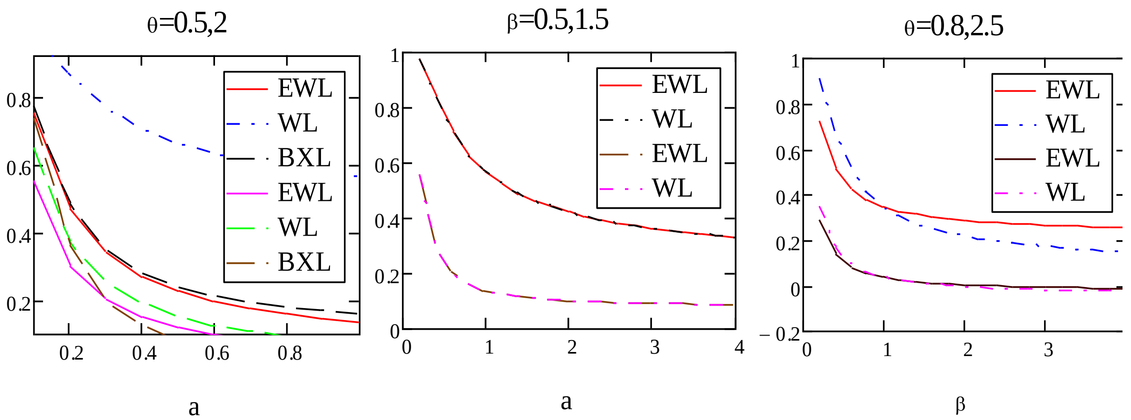
Furthermore, the variability of the skewness and kurtosis on the shape parameters β, θ and a can be investigated based on quantile measures. Bowley skewness based on quantile has been introduced by Kenney and Keeping (1962) and given by:

$$B = \frac{Q\left(\frac{3}{4}\right) - 2Q\left(\frac{1}{2}\right) + Q(\frac{1}{4})}{Q\left(\frac{3}{4}\right) - Q(\frac{1}{4})}$$

The Moors kurtosis (see Moors (1988)) based on quantiles is given by:

$$M = \frac{Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

where Q(.) denotes the quantile function. Plots of the skewness and kurtosis for some choices of the parameters θ and β and a are shown in Figures 3 and 4.

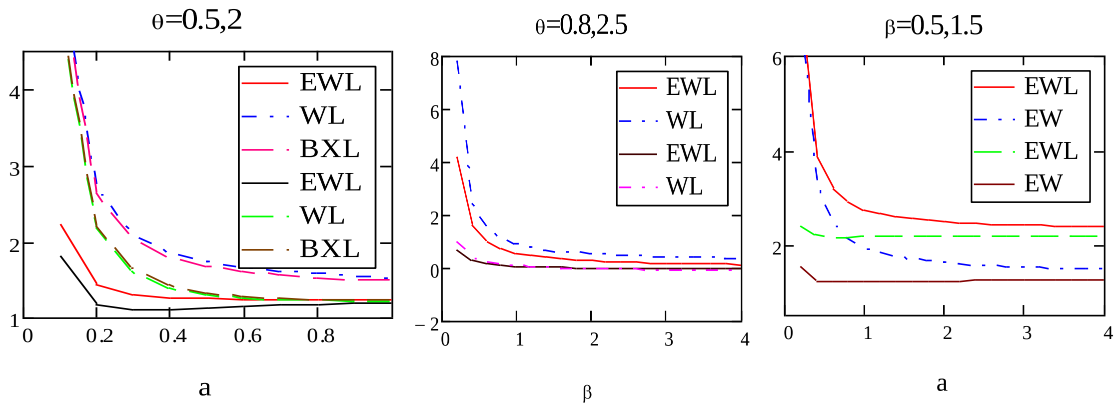


(i) Skewness for *EWL* as function of a and different values of θ

(ii) Skewness for *EWL* as function of a and different values of β

(iii) Skewness for *EWL* as function of β and different values of θ

Figure 3: Skewness plots for *EWL* distribution based on quantile function



(iv) Kurtosis for *EWL* as function of a and different values of θ

(v) Kurtosis for *EWL* as function of a and different values of β

(vi) Kurtosis for *EWL* as function of β and different values of θ

Figure 4 : Kurtosis plots for *EWL* distribution based on quantile function

It is obvious from Figure 3(i), that the skewness of *EWL* is smaller than the skewness for *WL* distribution at $\theta = 0.5$ and $\theta = 2.$ While the skewness of *EWL* and *WL* are equal at $\beta = 0.5, 1.5$ as seen in Figure 3(ii). The skewness of *EWL* is less than the skewness of *WL* at $\theta = 0.8$ as a function of β, while skewness of *EWL* is greater than the skewness of *WL* at $\theta = 2.5$ as a function of β (see Figure 3(iii)).

We can detect from Figures 4(iv) that the kurtosis of *EWL* is less than the kurtosis of *WL* at $\theta = 0.5, 2$ as a function of a. Also, the kurtosis of *EWL* is less than the kurtosis of *WL* at $\theta = 0.8, 2.5$ as a function of β (see Figure 4(vi)). While the kurtosis of *EWL* is greater than the kurtosis of *WL* at $\beta = 0.5, 1.5$ (see Figure 4(v)). Generally, these plots show that all the values of skewness and kurtosis decrease when the values of the parameters increase.

3.3 Rényi and q - Entropies

The entropy of a random variable fixed X is a measure of variation of uncertainty and has been used in many fields such as physics, engineering and economics. According to Rényi (1961), the Rényi entropy is defined by

$$I_\delta(x) = \frac{1}{1-\delta}\log\int_{-\infty}^{\infty} f(x)^\delta \, dx, \qquad \delta > 0 \; and \; \delta \neq 1$$

By applying the binomial theory (8), (10) and exponential expansion in the pdf (7), then the pdf $f(x;\Psi)^\delta$ becomes

$$(\mathrm{f}(x\,;\Psi))^\delta = \sum_{i,j,k=0}^{\infty} M_{i,j,k}\,(1+\frac{x}{\lambda})^{-\delta(\theta+1)}(1-(1+\frac{x}{\lambda})^{-\theta})^{\delta(\beta-1)+\beta j+k}$$

$$M_{i,j,k} = (-1)^{i+j}\binom{\delta(a-1)}{i}\frac{(a\alpha\theta\beta)^\delta(\alpha(\delta+i))^j\Gamma(\delta(\beta+1)+\beta j+k)}{\lambda^\delta k!\,\Gamma(\delta(\beta+1)+\beta j)}$$

Therefore, the Rényi entropy of *EWL* distribution is given by:

$$\mathrm{I}_\delta(\mathrm{x}) = \frac{1}{1-\delta}\log[\sum_{i,j,k=0}^{\infty} \mathrm{M}_{i,j,k}\frac{\lambda}{\theta}\mathrm{B}\left(\frac{\delta(\theta+1)-1}{\theta}, \delta(\beta-1)+\beta j+k+1\right)]$$

where $B(.,.)$ stands for beta function. Additionally, the q- entropy is defined by:

$$\mathrm{H_q(X)} = \frac{1}{1-q}\log\left(1-\int_{-\infty}^{\infty}\mathrm{f(x;\,\Psi)^q\,dx}\right), \mathrm{q} > 0 \text{ and } \mathrm{q} \neq 1$$

Therefore, the q- entropy of *EWL* distribution is given by:

$$\mathrm{H_q(X)} = \frac{1}{1-q}\log\{1-\sum_{i,j,k=0}^{\infty}\mathrm{M}_{i,j,k}\frac{\lambda}{\theta}\mathrm{B}\left(\frac{q(\theta+1)-1}{\theta}\right), q(\beta-1)+\beta j+k+1)]\}$$

3.4. Moments

The rth moment of *EWL* distribution can be obtained by using pdf (11) as follows

$$\mu_r' = \sum_{i,j,k=0}^{\infty} W_{i,j,k} \int_0^{\infty} x^r h_{\beta(j+1)+k}(x)\, dx$$

$$= \sum_{i,j,k=0}^{\infty} W_{i,j,k} \int_0^{\infty} x^r \frac{\theta(\beta(j+1)+k)}{\lambda} (1+\frac{x}{\lambda})^{-(\theta-1)} (1-(1+\frac{x}{\lambda})^{-\theta})^{\beta+\beta j+k-1}\, dx$$

Let $y = (1+\frac{x}{\lambda})^{-\theta}$ and using the binomial expansion, hence the *rth* moment of *EWL* distribution takes the following form:

$$\left\{ \begin{array}{l} \mu_r' = \sum_{i,j,k=0}^{\infty} \sum_{m=0}^{r} D_{i,j,k,m} (\beta(j+1)+k) \lambda^r \binom{r}{m} B\left(1-\left(\frac{1}{\theta}(r-m)\right), \beta(j+1)+k\right), \\ D_{i,j,k,m} = (-1)^m \binom{r}{m} W_{i,j,k} \end{array} \right\} \quad (15)$$

Setting $r = 1,2,3,4$ in (15), we can obtain the first four moments about zero. Furthermore, the moment generating functionof *EWL* distribution is obtained as follows:

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r'$$

$$= \sum_{i,j,k=0}^{\infty} \sum_{m=0}^{r} D_{i,j,k,m} (\beta(j+1)+k) \lambda^r \frac{t^r}{r!} \binom{r}{m} B(1-\left(\frac{1}{\theta}(r-m)\right), B(j+1)+k)$$

3.5 The Probability Weighted Moments

The probability weighted moments (PWM) of a random variable *X* following the *EWL* distribution, say $\tau_{r,s}$, is formally defined by:

$$\tau_{r,s} = E[X^r F(x)^s] = \int_{-\infty}^{\infty} x^r f(x) (F(x))^s\, dx \quad (16)$$

Inserting (11) and (13) into (16), hence, the PWM of *EWL* distribution takes the following form

$$\tau_{r,s} = \sum_{i,j,k,p,q,l=0}^{\infty} \sum_{m=0}^{r} (-1)^r W_{i,j,k} \eta_{p,q} (\beta(j+1)+k) \lambda^r B(1-\frac{r-m}{\theta}, \beta q+1+\beta+\beta j+k)$$

3.6 Moments of Residual Life Function

The residual life plays an important role in life testing situations and reliability theory. The *n*th moment of the residual life is given by:

$$m_n(t) = E[(X-t)^n | X > t] = \frac{1}{R(t)} \int_t^{\infty} (x-t)^n f(x)\, dx \qquad (17)$$

The *n*th moment of the residual life of a random variable has *EWL* distribution is obtained by inserting pdf (11) in (17) as follows

$$\left\{ \begin{array}{c} m_n(t) = \dfrac{\sum_{r=0}^{n} \sum_{m=0}^{r} \sum_{i,j,k}^{\infty} N_{i,j,k,r,m} \lambda^r Beta[(1+\frac{t}{\lambda})^{-\theta}, 1-\left(\frac{1}{\theta}(r-m)\right), \beta(j+1)+k]}{1-[1-\exp(-\alpha((1+\frac{t}{\lambda})^{\theta}-1)^{\beta})]^{a}}, \\ N_{i,j,k,r,m} = (-1)^{n-r+m} W_{i,j,k} \binom{r}{m} \binom{n}{r} t^{n-r} (\beta(j+1)+k) \end{array} \right\} \qquad (18)$$

Another interesting function is the mean residual life (MRL*)*,which represents the expected additional life length for a unit which is alive at age x . The *MRL* of the EWL distribution is obtained by putting $n=1$ in (18).

3.7 Order Statistics

Order statistics play a vital role in many areas of statistical theory and practice. We derive an explicit expression for the density function of the *r*th order statistic $X_{r:n}$ in a random sample $X_{1:n} < X_{2:n} < \ldots < X_{n:n}$, of size *n* from the *EWL* distribution. The pdf of *r*th order statistics can be written as follows

$$f_{r,n}(x) = \frac{f(x)}{B(r, n-r+1)} \sum_{v=0}^{n-r} (-1)^v \binom{n-r}{v} F(x)^{v+r-1} \qquad (19)$$

Inserting (11) in (19) and (13) by replacing s with $v+r-1$, in (19), leads to

$$f_{r,n}(x) = \sum_{v=0}^{n-r} \sum_{i,j,k,p,q,l}^{\infty} K_{i,j,k,p,q,l} h_{\beta+\beta j+\beta q+l+k}(x),$$

$$K_{i,j,k,p,q} = (-1)^{v+p+q} \binom{n-r}{v} \binom{a(v+r-1)}{p} \frac{(\alpha p)^q \Gamma(\beta q+1)}{q!\,1!\,\Gamma(\beta q)} \frac{W_{i,j,k}}{B(r,n-r+1)} \frac{\Gamma(\beta(j+1)+k+1)}{\beta+\beta j+\beta q+1+k} \qquad (20)$$

Therefore, the pdf of *r*th order statistics of Exponentiated Weibull Lomax can be expressed as a mixture of Exponentiated Lomax densities with parameters θ, λ and $\beta j + \beta + \beta q + \ell + k$.

Further, the pdf of the smallest order statistics is obtained by subsituting $r=1$ in (20) as follows

$$f_{1:n}(x)=\sum_{v=0}^{n-1}\sum_{i,j,k,p,q,l}^{\infty}\varsigma_{i,j,k,p,q,l}h_{\beta+\beta j+\beta q+l+k}(x),$$

$$\varsigma_{i,j,k,p,q}=(-1)^{v+p+q}\binom{n-1}{v}\binom{av}{p}W_{i,j,k}\frac{n(\alpha p)^{q}\Gamma(\beta q+1)}{q!\,1!\,\Gamma(\beta q)}\frac{\Gamma(\beta(j+1)+k+1)}{\beta+\beta j+\beta q+1+k}$$

Also, the pdf of the largest order statistics is obtained by subsituting $r=n$ in (20) as follows

$$f_{n:n}(x)=\sum_{i,j,k,p,q,l}^{\infty}\omega_{i,j,k,p,q,l}h_{\beta+\beta j+\beta q+l+k}(x),$$

$$\omega_{i,j,k,p,q}=(-1)^{p+q}W_{i,j,k}\frac{n(\alpha p)^{q}\Gamma(\beta q+1)}{q!\,1!\,\Gamma(\beta q)}\frac{\Gamma(\beta(j+1)+k+1)}{\beta+\beta j+\beta q+1+k}$$

4. Different Estimation Methods

This section concerns with the point estimates of the model parameter for *EWL* distribution using three different methods. The maximum likelihood estimators, least squares estimators and weighted least squares estimators are derived in the following subsections.

4.1 Maximum Likelihood Estimators

The maximum likelihood (ML) estimators of the unknown parameters for the expoentiated Weibull Lomax distribution are obtained. Let $X_1,...,X_n$ be observed values from the *EWL* distribution with set of parameters $\Psi=(a,\alpha,\theta,\beta,\lambda)^T$. The log-likelihood function for the vector of parameters Ψ can be written as

$$lnL(\Psi)=nlna+nln\alpha+nln\beta+nln\theta-nln\lambda+(\beta+1)\sum_{i=1}^{n}ln\left(1-Zi^{-\theta}\right)+(\theta\beta-1)\sum_{i=1}^{n}lnZi-\alpha[Zi^{-\theta}-1]^{\beta}+(a-1)\sum_{i=1}^{n}ln[1-exp(-\alpha[Zi^{-\theta}-1]^{\beta})]$$

where, $z_i=\left(1+\frac{x_i}{\lambda}\right)$. The elements of the score function $\mathrm{U}(\Psi)=(U_a,U_\alpha,U_\beta,U_\theta,U_\lambda$ are given by

$$U_a = \frac{n}{a} + \sum_{i=1}^{n} \ln[1 - \exp(-\alpha[Zi^{-\theta} - 1]^{\beta})] \circ$$

$$U_a = \frac{n}{a} - [Zi^{-\theta} - 1]^{\beta} + (a-1)\sum_{i=1}^{n} \frac{\exp(-\alpha[Zi^{-\theta}-1]^{\beta})[Zi^{-\theta}-1]^{\beta}}{[1-\exp(\alpha[Zi^{-\theta}-1]^{\beta})]},$$

$$U_\beta = \frac{n}{\beta} + \sum_{i=1}^{n} \ln(1 - Zi^{-\theta}) + \theta \sum_{i=1}^{n} lnZi - \alpha[\,Zi^{\theta} - 1]^{\beta} \ln[Zi^{\theta} - 1] + \alpha(a - 1)\sum_{i=1}^{n} \frac{\exp(-\alpha[Zi^{\theta} - 1]^{\beta}[Zi^{\theta} - 1]^{\beta} ln[Zi^{\theta} - 1]}{[1 - \exp(-\alpha[Zi^{\theta} - 1]^{\beta})]}$$

$$U_\lambda = -\frac{n}{\lambda} - \frac{\theta(\beta+1)}{\lambda^2}\sum_{i=1}^{n} \frac{x_i {Z_i}^{-\theta-1}}{(1-Zi^{-\theta})} - \frac{(\theta\beta - 1)}{\lambda^2}\sum_{i=1}^{n} \frac{x_i}{Z_i} + \frac{\alpha\beta\theta}{\lambda^2}[Zi^{\theta} - 1]^{\beta-1} Zi^{\theta-1} x_i - \frac{(\alpha-1)\alpha\beta\theta}{\lambda^2}\sum_{i=1}^{n} \frac{\exp(-\alpha[Zi^{\theta} - 1]^{\beta})[Zi^{\theta} - 1]^{\beta-1} Zi^{\theta-1} x_i}{[1 - \exp(-\alpha[Zi^{\theta} - 1]^{\beta})]},$$

$$U_\theta = \frac{n}{\theta} + (\beta+1)\sum_{i=1}^{n} \frac{lnZ_i}{({Z_i}^{\theta} - 1)} + \beta \sum_{i=1}^{n} ln\, Z_i - \alpha\beta[{Z_i}^{\theta} - 1]^{\beta-1} {Z_i}^{\theta} lnZ_i + (a - 1)\sum_{i=1}^{n} \frac{exp(-\alpha[Zi^{\theta} - 1]^{\beta})\alpha\beta[{Z_i}^{\theta} - 1]^{\beta-1} {Z_i}^{\theta} lnZ_i}{[1 - exp(\alpha[Zi^{\theta} - 1]^{\beta})},$$

ML estimators of the model parameters are determined by solving numerically the non-linear equations $U_a = 0, U_\alpha = 0, U_\beta = 0, U_\theta = 0$ and $U_\lambda = 0$ simultaneously by using mathematical package.

4.2 Least Squares Estimators

Suppose $X_1, ..., X_n$ is a random sample of size n from *EWL* distribution and suppose $X_{1:n} < X_{2:n} < ... < X_{n:n}$ denotes the corresponding ordered sample. According to Johnson et al. (1995), the expectation and the variance of distribution are independent of the unknown parameter and are given by

$$E(F(X_{i:n})) = \frac{i}{n+1}\ , var(F(X_{i:n})) = \frac{i(n-i+1)}{(n+1)^2(n+2)}$$

where $F(X_{i:n})$ is cdf for any distribution and $X_{i:n}$ is the *i*th order statistic. Hence, the least squares (LS) estimators can be obtained by minimizing the sum of squares errors

$$\sum_{i=1}^{n} (F(X_{i:n}) - \frac{i}{n+1})^2,$$

with respect to the unknown parameters. So the LS estimators of the unknown parameters a, α, θ, β and λ are obtained by minimizing the following quantity

$$\sum_{i=1}^{n}([1-\exp(\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta}]^{a}-\frac{i}{n+1})^{2},$$

with respect to a, α, θ, β and λ. It is very hard to obtain a closed form solution, so mathematical software will be applied.

4.3 Weighted Least Squares Estimators

Here the weighted least squares (WLS) estimators of the unknown parameters for *EWL* are derived. Again, let $X_1, ..., X_n$ is a random sample of size n from *EWL* distribution and $X_{1:n} < X_{2:n} < ... < X_{n:n}$ be the corresponding ordered sample. The WLS estimators can be obtained by minimizing the following sum of squares errors

$$\sum_{i=1}^{n}\frac{1}{var(F(X_{i:n}))}\left[F(X_{i:n})-E\big(F(X_{i:n})\big)\right]^{2},$$

with respect to the unknown parameters α, λ, θ and β. Therefore, the WLS estimators will be obtained by minimizing the following quantity

$$\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}([1-\exp(-\alpha((1+\frac{x}{\lambda})^{\theta}-1)^{\beta}]^{a}-\frac{i}{n+1})^{2},$$

with respect to a, α, θ, β and λ.

5. Simulation Study

In this section, an extensive simulation study is conducted to compare the performance of the different estimators in the sense of their relative biases (RBs) and standard errors (SEs) for different sample sizes and for different parameter values.1000 samples of small, moderate and large sample sizes are generated from *EWL* distribution with different set of parameters. Without loss of generality, we take the scale parameter λ to be known and equal one throughout the experiment and six sets of parameters are considered. The RBs and SEs of the ML, LS and WLS estimates of the models parameters are listed in Tables (1, 2 and 3). The simulation study is carried out as follows:

Step 1: Generate 1000 random samples of size 10, 20, 30, 50 and 100 from the *EWL* distribution.

Step 2: Six sets of parameters values are selected as; case 1 ≡ (α = 0.5, a = 1.5, θ = 0.5, β = 0.5), case 2 ≡ (α = 0.5, a = 1.5, θ = 0.5, β = 1.5), case 3 ≡ (α = 0.5, a = 1.5, θ = 1.5, β = 0.5), case 4 ≡ (α = 0.5, a = 1.5, θ = 1.5, β = 1.5)), case 5≡ (α = 0.5, a = 0.8, θ = 0.5, β = 0.8)) and case 6 ≡ (α = 0.5, a = 0.8, θ = 0.5, β = 0.5)

Step 3: The ML, LS and WLS estimates of the unknown parameters are obtained.
Step 4: The RBs and SEs of different estimates of unknown parameters are computed.

All the results of the simulation are listed in Tables (1, 2, and 3). Some conclusion can be deducted about the performance of different estimators:

1. For all different values of estimates and different methods of estimation we can realize that the SEs decrease as sample size increases. (see Tables (1, 2, and 3)).

2. The SEs of ML estimates, for all parameters values, are the largest among the other estimates in all cases (see Tables (1, 2, and 3)).

3. Based on Tables 1, 2 and 3, the SEs for the β estimate increase as the value of the parameter β increases, for all different methods of estimation.

4. The SEs of the estimate θ are the smallest for set of parameters 2 and 4, for different methods of estimation and different sample sizes (see Tables 1 and 2).

5. Depending on Tables 1 and 2, both the RBs and SEs for θ decrease when the value of θ increases.

Table 1: Results of simulation study of RBs and SEs of estimates for different values of parameters $(\alpha, a, \theta, \beta)$ for the Exponentiated Weibull Lomax distribution

n	Method	Properties	Case 1				Case 2			
			$\alpha = 0.5$	$a = 1.5$	$\theta = 0.5$	$\beta = 0.5$	$\alpha = 0.5$	$a = 1.5$	$\theta = 0.5$	$\beta = 1.5$
10	ML	RB	0.808	0.425	0.141	1.241	1.310	1.209	0.095	0.277
		SE	0.145	0.321	0.035	0.068	0.176	0.447	0.025	0.079
	LS	RB	0.266	0.27	0.130	0.616	0.232	0.047	0.045	0.129
		SE	0.082	0.085	0.028	0.026	0.065	0.129	0.011	0.055
	WLS	RB	0.294	0.258	0.113	0.594	0.290	0.073	0.047	0.136
		SE	0.083	0.087	0.031	0.027	0.073	0.138	0.013	0.057
20	ML	RB	0.122	0.146	0.182	1.023	1.049	0.826	0.021	0.156
		SE	0.041	0.086	0.015	0.026	0.068	0.163	9.668*	0.034
	LS	RB	8.03*	0.339	0.182	0.639	0.174	6.912*	0.065	0.123
		SE	0.031	0.033	0.011	9.592*	0.024	0.051	3.452*	0.023
	WLS	RB	0.028	0.333	0.19	0.621	0.167	0.012	0.071	0.133
		SE	0.029	0.029	0.011	9.991*	0.02	0.042	3.746*	0.023
30	ML	RB	0.057	0.323	0.267	1.003	0.583	0.412	7.859	0.129*
		SE	0.019	0.031	6.883*	0.014	0.031	0.07	4.505*	0.018
	LS	RB	0.08	0.367	0.248	0.668	0.046	0.062	0.051	0.131
		SE	0.016	0.016	5.062*	5.063*	0.011	0.024	2.164*	0.014
	WLS	RB	0.07	0.359	0.249	0.655	0.091	0.051	0.063	0.125
		SE	0.015	0.016	5.508*	5.578*	9.777*	0.021	2.26*	0.013
50	ML	RB	0.135	0.377	0.288	0.869	0.361	0.214	0.024	0.114
		SE	7.642*	0.013	5.217*	5.727*	0.014	0.029	3.142*	8.684*
	LS	RB	0.144	0.394	0.282	0.693	5.407*	0.09	0.051	0.123
		SE	7.008*	6.526*	2.461*	2.6*	3.746*	9.335*	1.217*	6.985*
	WLS	RB	0.125	0.361	0.276	0.632	0.064	0.065	0.063	0.118
		SE	6.055*	6.277*	2.556*	6.095*	4.095*	9.421*	2.188*	6.789*
100	ML	RB	0.255	0.445	0.337	0.833	0.135	0.031	0.04	0.103
		SE	2.234*	2.661*	4.184*	1.352*	3.657*	7.26*	2.547*	3.275*
	LS	RB	0.224	0.409	0.303	0.702	0.037	0.099	0.037	0.115

		SE	2.522*	2.571*	2.061*	1.159*	8.975*	3.056*	5.921*	3.051*
	WLS	RB	0.214	0.384	0.294	0.653	4.577*	0.085	0.044	0.108
		SE	2.051*	2.389*	2.402*	1.426*	3.963*	2.995*	1.663*	2.889*

* Indicate that the value multiply 10^{-3}

Table 2: Results of simulation study of RBs and SEs of estimates for different values of parameters $(\alpha, a, \theta, \beta)$ for the Exponentiated Weibull Lomax distribution.

n	Method	Properties	Case 3				Case 4			
			$\alpha = 0.5$	$a = 1.5$	$\theta = 1.5$	$\beta = 0.5$	$\alpha = 0.5$	$a = 1.5$	$\theta = 1.5$	$\beta = 1.5$
10	ML	RB	0.753	0.336	0.411	3.065	3.405	2.042	0.367	1.855
		SE	0.152	0.385	0.066	0.158	0.278	0.669	0.033	0.395
	LS	RB	0.875	0.253	0.454	1.267	2.111	0.353	0.303	0.538
		SE	0.129	0.128	0.053	0.060	0.150	0.217	0.04	0.126
	WLS	RB	0.869	0.245	0.422	1.16	2.235	0.420	0.296	0.553
		SE	0.133	0.130	0.54	0.057	0.169	0.242	0.041	0.134
20	ML	RB	0.253	0.254	0.453	2.298	2.108	0.896	0.316	1.072
		SE	0.041	0.082	0.029	0.055	0.088	0.213	0.017	0.137
	LS	RB	0.468	0.383	0.486	1.333	1.763	0.171	0.322	0.543
		SE	0.039	0.038	0.022	0.027	0.055	0.086	0.014	0.061
	WLS	RB	0.393	0.378	0.434	1.207	1.747	0.195	0.3	0.497
		SE	0.038	0.036	0.023	0.025	0.058	0.085	0.016	0.059
30	ML	RB	0.029	0.405	0.459	2.117	1.566	0.438	0.303	0.801
		SE	0.019	0.035	0.015	0.032	0.039	0.101	9.194*	0.069
	LS	RB	0.345	0.432	0.499	1.419	1.528	0.074	0.308	0.55
		SE	0.023	0.021	0.014	0.017	0.030	0.048	9.139*	0.039
	WLS	RB	0.274	0.415	0.434	1.214	1.454	0.088	0.291	0.487
		SE	0.024	0.021	0.015	0.016	0.03	0.047	8.667*	0.038
50	ML	RB	0.134	0.517	0.471	1.886	1.325	0.197	0.291	0.475
		SE	7.767*	0.011	6.87*	0.015	0.016	0.037	4.478*	0.024
	LS	RB	0.137	0.503	0.516	1.54	1.425	0.022	0.319	0.497
		SE	8.223*	8.302*	7.323*	9.856*	0.014	0.023	4.003*	0.022
	WLS	RB	0.036	0.479	0.435	1.255	1.302	0.02	0.294	0.439
		SE	8.387*	7.709*	8.096*	9.088*	0.014	0.022	4.095*	0.021
100	ML	RB	0.222	0.562	0.490	1.735	1.028	0.094	0.299	0.405
		SE	2.425*	2.976*	2.841*	5.847*	3.944*	7.658*	4.469*	7.377*
	LS	RB	0.037	0.569	0.557	1.751	1.09	0.138	0.308	0.509
		SE	2.15*	2.732*	3.142*	4.663*	3.615*	6.875*	*4.606**	*0.01*
	WLS	RB	0.097	0.515	0.443	1.286	0.948	0.094	*0.273*	*0.372*
		SE	2.481*	2.547*	3.784*	4.848*	4.089*	6.418*	3.577*	8.88*

* Indicate that the value multiply 10^{-3}

Table 3: Results of simulation study of RB*s* and SE*s* of estimates for different values of parameters $(\alpha, a, \theta, \beta)$ for the Exponentiated Weibull Lomax distribution.

n	Method	Properties	Case 5				Case 6			
			$\alpha = 0.5$	$a = 0.8$	$\theta = 0.5$	$\beta = 0.8$	$\alpha = 0.5$	$a = 0.8$	$\theta = 0.5$	$\beta = 0.5$
10	ML	RB	1.101	1.828	0.263	0.67	0.756	1.183	0.148	1.029
		SE	0.156	0.356	0.0489	0.109	0.126	0.296	0.045	0.073
	LS	RB	0.900	0.441	0.204	0.039	0.661	0.120	0.196	0.154
		SE	0.118	0.111	0.045	0.04	0.096	0.070	0.050	0.028
	WLS	RB	0.996	0.536	0.236	5.443*	0.703	0.168	0.179	0.142
		SE	0.123	0.128	0.047	0.038	0.097	0.079	0.049	0.027
20	ML	RB	0.472	0.757	0.149	0.473	0.241	0.235	0.023	0.821
		SE	0.047	0.108	0.018	0.043	0.036	0.064	0.015	0.03
	LS	RB	0.592	0.272	0.123	0.048	0.424	0.063	0.091	0.145
		SE	0.045	0.041	0.017	0.017	0.037	0.027	0.019	0.012
	WLS	RB	0.650	0.343	0.187	0.022	0.422	0.058	0.054	0.155
		SE	0.046	0.043	0.019	0.015	0.035	0.025	0.018	0.012
30	ML	RB	0.273	0.297	0.075	0.397	0.084	0.062	0.029	0.676
		SE	0.022	0.038	9.975*	0.025	0.018	0.031	9.91*	0.016
	LS	RB	0.394	0.143	0.039	0.077	0.198	0.018	0.091	0.136
		SE	0.023	0.02	9.016*	0.01	0.017	0.011	0.011	7.186*
	WLS	RB	0.421	0.215	0.135	0.016	0.203	3.598*	0.085	0.125
		SE	0.023	0.020	0.01	9.362*	0.015	0.011	0.010	7.116*
50	ML	RB	0.249	0.194	0.028	0.228	0.022	0.094	0.117	0.610
		SE	9.722*	0.015	4.628*	0.01	8.36*	0.012	4.394*	7.766*
	LS	RB	0.293	0.056	5.637*	0.088	0.163	0.021	0.024	0.153
		SE	9.785*	7.71*	4.728*	5.393*	7.851*	5.848*	5.24*	4.172*
	WLS	RB	0.328	0.167	0.106	0.036	0.17	4.22*	0.023	0.126
		SE	9.214*	7.377*	5.196*	5.163*	6.969*	5.543*	4.877*	4.036*
100	ML	RB	0.117	0.032	0.02	0.163	0.109	0.214	0.183	0.531
		SE	3.027*	3.945*	1.728*	3.208*	2.822*	3.999*	1.514*	2.828*
	LS	RB	0.132	0.014	0.046	0.132	0.056	0.027	0.013	0.15
		SE	3.303*	2.954*	1.628*	2.516*	2.464*	2.501*	1.852*	1.985*
	WLS	RB	0.168	0.119	0.09	0.024	0.072	8.021*	0.011	0.106
		SE	2.866*	2.67*	2.089*	2.547*	2.217*	2.372*	1.806*	1.918*

* Indicate that the value multiply 10^{-3}

6. Applications to Real Data

In this section, two real data sets are provided to illustrate the importance and flexibility of *EWL* distribution comparing with main four models; Exponentiated generalized modified Weibull *(EGMW)* (Aryal and Elbatal (2015)), Beta modified Weibull (*BMW*) (Silva et al. (2010)),Kumaraswamy Lomax (*KL*); and Weibull Lomax (*WL).* The method of maximum likelihood is used to estimate the unknown parameters of the selected models. The following statistics: -2log-likelihood function $(-2\ln \ell)$ evaluated at the parameter estimates, Akaike information criterion (AIC), the corrected Akaike information criterion (CAIC), the Hannan-Quinn information criterion (HQIC), Anderson-Darling (A^*) criterion and Cramér–von Mises (W^*) criterion are used to compare all the models. However, the better distribution corresponds to the smaller values of AIC, BIC, CAIC, HQIC, A^* and W^* criteria. Furthermore, we plot the histogram for each data set and the estimated pdf of the models. Moreover, the plots of empirical cdf of the data sets and estimated cdf of the models are displayed in Figures 5 and 6.

Data Set 1:

The real data represents 34 observations of the vinyl chloride data obtained from clean up gradient ground –water monitoring wells in mg/L. The data are obtained from Bhaumik et al. (2009) and recorded as follows

Models	Estimated Parameters						
	α	γ	λ	θ	β	δ	ε
EWL	5.158 (3.631)	86.306 (366.672)	58.308 (236.529)	25.057 (94.904)	0.158 (0.169)	-	-
KL	-	-	0.461 (1.747)	0.04 (0.378)	-	1.35 (0.919)	163.759 (2015.539)
WL	-	-	0.019 (0.15)	0.766 (10.305)	-	36.64 (930.774)	0.996 (0.198)
BMW	0.561 (14.561)	0.778 (0.611)	6.87×10^{-5} (0.155)	-	-	1.615 (2.219)	1.668 (47.502)
EGMW	0.036 (0.00854)	0.0084 (17.699)	0.913 (4.76)	15.369 (0.462)	1.076 (2.578)		

5.1, 1.2, 1.3, 0.6, 0.5, 2.4, 0.5, 1.1, 8.0, 0.8, 0.4, 0.6, 0.9, 0.4, 2.0, 0.5, 5.3, 3.2, 2.7, 2.9, 2.5, 2.3, 1.0, 0.2, 0.1, 0.1, 1.8, 0.9, 2.0, 4.0, 6.8, 1.2, 0.4,0.2.

Table 4: ML estimates and their SEs (in parentheses) for the first data set

Table (5) Model selection criteria for the first data set

Model	$-2\ln\ell$	AIC	CAIC	BIC	HQIC	A^*	W^*
EWL	108.642	118.642	120.642	126.274	121.245	0.1652	0.0232
EGMW	110.804	120.804	122.949	128.435	123.406	0.2882	0.0444
BMW	110.191	120.191	122.333	127.822	122.793	0.2317	0.0356
KL	129.527	137.527	138.817	143.632	139.609	0.2025	0.0304
WL	128.325	136.325	137.615	142.43	138.407	0.3173	0.0487

Results in Table 5, indicate that the *EWL* model is more suitable than the other competitive models for this data set based on the selected criteria. Additionally, it is clear from Figure 5 that the *EWL* distribution provides a better fit and therefore be one of the best models for this data set.

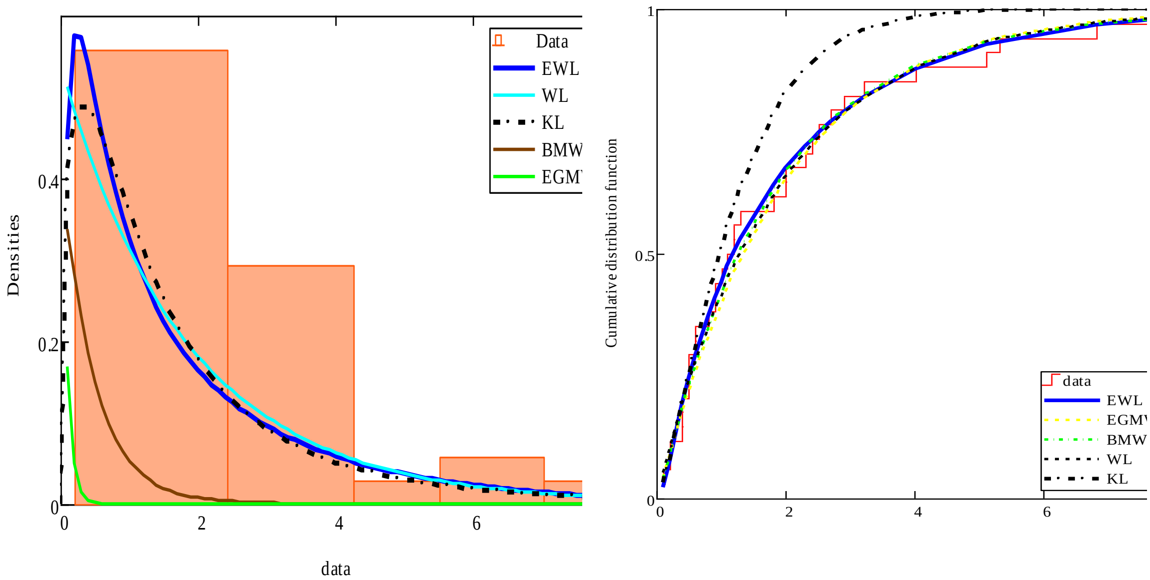


Figure 5: Densities and distributions of models for the first data set

Data Set 2:

The second real data set corresponds to an uncensored data set from Nichols and Padgett (2006) on breaking stress of carbon fibres (in Gba). The data are recorded as follows

3.70,2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19,3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39,2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83,1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00,1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65.

Results in Table 7, show that the *EWL* model is more suitable than the other competitive models for this data set based on the above selected criteria.

Table 6: ML estimates and their SEs (in parentheses) for the second data set

Models	Estimated Parameters						
	α	γ	λ	θ	β	δ	ε
EWL	0.03 (0.639)	1.665 (4.274)	0.742 (8.259)	1.601 (6.356)	1.563 (5.67)	-	-
WL	-	-	0.336 (1.224)	1 (1.043)	-	1.032 (11.506)	2.769 (0.874)
BMW	7.132 (5.915)	2.199 (0.694)	0.207 (0.23)			0.341 (0.223)	6.687×10^{-3} (5.599×10^{-3})
EGMW	0.021 (2.243×10^{-3})	1.582 (0.768)	3.032 (1.976)	1.363×10^{-3} (0.383)	0.892 (5.843×10^{-3})		

Table (7) Model selection criteria for the second data set

Model	$-2\ln\ell$	AIC	CAIC	BIC	HQIC	A^*	W^*
EWL	282.624	292.624	293.249	305.65	297.896	0.4125	0.0706
WL	457.607	465.607	466.019	476.028	469.824	153.316	9.813
BMW	287.823	297.823	298.461	310.849	303.095	86.216	17.183
EGMW	283.544	293.544	294.183	306.57	298.816	99.482	3.950

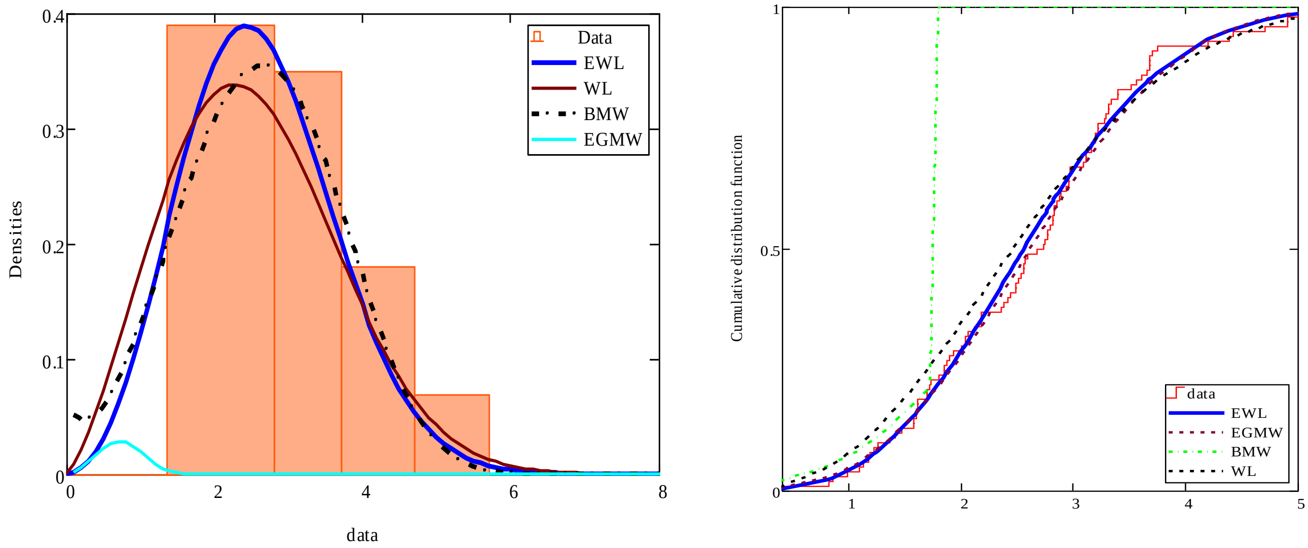


Figure 6: Densities and distributions of models for the second data set

Additionally, it is clear from Figure 6 that the *EWL* distribution provides a better fit than the *EGMW*, *BMW* and *WL* for this data set.

7. Conclusion

In this paper, we present a new class of distributions, called the Exponentiated Weibull Lomax, based on Exponentiated Weibull-G family. The *EWL* distribution generalizes the Weibull Lomax distribution presented by Tahir et al. (2016 a) and at the same time, provides some new models. Some properties of the *EWL* distribution such as, moments, mean residual life, order statistics, quantile, Re'nyi and q- entropies are derived. The maximum likelihood, least squares, and weighted least squares estimators are obtained and simulation study is provided to compare the model performance of the estimates. An application of the *EWL* distribution to two real data sets show that the new distribution can be used quite effectively to provide better fits than Kumaraswamy-Lomax, Weibull-Lomax, beta modified Weibull and Exponentiated generalized modified Weibull models.

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