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THE GENERALIZED WEIBULL-BURR XII DISTRIBUTION AND ITS APPLICATIONS

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Abstract: In this paper, we introduce a new lifetime model, called the Gen- eralized Weibull-Burr XII distribution. We discuss some of its mathematical properties such as density, hazard rate functions, quantile function and mo- ments. Maximum likelihood method is used to estimate model parameters. A simulation study is performed to assess the performance of maximum like- lihood estimators by means of biases, mean squared errors. Finally, we prove that the proposed distribution is a very competitive model to other classical models by means of application on real data set.

Key words: Burr XII distribution, Maximum Likelihood estimation,Weibull- X family, Moment.

1. Introduction

There has been a great interest in developing more flexible distributions through extending the classical distributions by introducing additional shape parameters to the baseline model. Many generalized families of distributions have been pro- posed and studied over the last two decades for modeling data in many applied areas such as economics, engineering, biological studies, environmental sciences, medical sciences and finance. Some well-known families can be cited as follows: Beta-G by Eugene et al.(2002) and Jones(2004), gamma-G type 1 by Zografos and Balakrishnan(2009), Kumaraswamy-G (Kw-G) by Cordeiro and Castro(2011), McDonald-G(Mc-G) by Alexander et al. (2012), Gamma-G type 2 by Ristic and Balakrishnan (2012), Gamma-G type 3 by Torabi and Montazeri (2012), Expo- nentiated Generalized-G by Cordeiro et al. (2013), Exponentiated T-X family by Alzaghal et al.(2013), Gamma-X by Alzaatreh et al.(2013), Logistic-G by Torabi and Montazeri (2014), Log-gamma-G by Amini et al. (2014), T-X {Y } family based on quantile function approach by Aljarrah et al.(2014), Weibull- G by Bourguignon et al.(2014), Exponentiated Half-logistic-G by Cordeiro et al.(2014), T-R{Y } family by Alzaatreh et al.(2014), Lomax-G by Cordeiro et al.(2014), Kumaraswamy odd log-logistic-G by Alizadeh et al.(2015), logistic-X by Tahir et al.(2016) , Weibull-G by Tahir et al.(2016), Generalization of Burr XII distribution by Mead (2014), McDonald Burr XII by Antonio et al.(2014), Generalization gamma distribution by Alzaatreh et al.(2015) , generalized normal distribution by Korkmaz and Genc (2017),Beta BXII by Paranaba et al.(2013) and Kumaraswamy BXII by Paranaba et al.(2013).

Let r(t) be the probability density function (pdf) of a random variable Tϵ[a, b] for $-\infty < a < b < \infty$ and let W[G(x)] be a function of the cumulative distri- bution function (cdf) of a random variable X such that W[G(x)] satisfies the following three conditions: (i) W [G(x)] ϵ [a,b], (ii) W [G(x)] is differentiable and monotonically non-decreasing, (iii) W [G(x)] $\rightarrow$ a as x $\rightarrow -\infty$ and W[G(x)] $\rightarrow$ b as x$\rightarrow \infty$

Alzaatreh et al. (2013) defined the T-X family of distributions by

$$F(x) = \int_0^{W[G(x)]} r(t)dt.$$

where W[G(x)] satisfies the (i), (ii) and (iii), the pdf corresponding to F (x) is given by

$$f(x) = \{\frac{d}{dx}W[G(x)]\}r\{W[G(x)]\}.$$

Let T follows the Weibull distribution with parameters α and β , $r(t) = \alpha\beta t^{\beta-1}e^{-\alpha t\beta}$, $t \geq 0$ and $W[G(x)] = -\log[1 - G(x)]$, Cordeiro et al. (2015) introduced the generalized Weibull family (GW-G) of distributions by cdf

$$F(x) = \alpha\beta \int_0^{-\log[1-G(x)]} t^{\alpha-1}\mathrm{e}^{-\alpha t^{\beta}}\mathrm{dt} \qquad (1)$$

and pdf

$$f(x) = \frac{\alpha\beta g(x)}{[1 - G(x)]}\{-\log[1 - G(x)]\}^{\beta-1} \exp\{-\alpha(-\log[1 - G(x)])^{\beta}\}. \qquad (2)$$

where α > 0 and β > 0 are the additional shape parameters.

The main aim of this paper we introduce a new generalization of the Burr XII distribution called the Weibull-Burr XII distribution by using the Weibull-G family introduced by Alzaatreh et al. (2013) and Cordeiro et al(2015). Burr XII distribution is widely used in quality control, technical defect data, the perfor- mance of housing in medical experiments and reliability systems. It is desired to provied more flexible density and hazard rate compared to other existing Burr generalization. The data admit the concept.

The rest of the paper is organized as follows: In Sec.2, Generalized Weibull- Burr XII distribution is defined. In Sec.3, some mathematical properties of pro- posed distribution are obtained. In Section 4, skewness and kurtosis measures of the proposed model are discussed. In Sec.5, we provide on expansion for its density function as a mixture of Burr XII densities. In Sec.6 and Sec.7, the moment and moment generating functions are obtained. The model parameters are estimated by maximum likelihood and a simulation study is performed in In Sec.8 and Sec.9. In Sec.10, Application of the Weibull-Burr XII to real data set is provided. Finally, some conclusions and future work are noted in Section 11.

2. The Weibull-Burr XII Distribution

Let baseline distribution be the Burr XII distribution(with parameters s; k; u) with pdf and cdf respectively, given by $g(x) = uk s^{-u} x^{u-1}[1 + (x/s)^u]^{-k-1}$ and $G(x) = 1 - [1 + (x/s)^u]^{-k}$, $s, k, u > 0$.

Then, pdf of Generalized Weibull-Burr XII (GWBXII) distribution is given by

$$f(x) = \frac{\alpha\beta u k s^{-u} x^{u-1}}{[1+(\frac{x}{s})^u]}\{-\log[1+(\frac{x}{s})^u]^{-k}\}^{\beta-1}\exp\{-\alpha(-\log[1+(\frac{x}{s})^u]^{-k})^{\beta}\},\ \ x>0. \tag{3}$$

where $\alpha > 0$, β, $s > 0$, $k > 0$, $u > 0$ and denoted by GWBXII (α,β,s,k,u). The cdf, corresponding to (3) is given by

$$F(x) = 1 - \exp\{-\alpha(-\log[1+(\frac{x}{s})^u]^{-k})^{\beta}\}. \tag{4}$$

Plots of the pdf and hrf of the GWBXII distribution for the selected parameter values are shown in Figure 1 and 2. These figures show that the pdf of the GWBXII is right-skewed and nearly symmetric and its hrf can be monotonically increasing or decreasing depending basically on the parameter values.

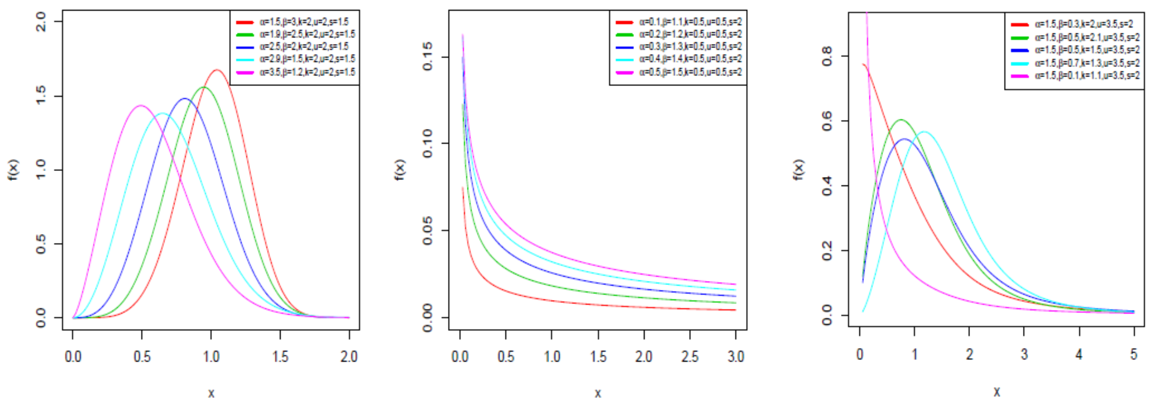

Figure 1: Plots of pdf of the GWBXII distribution for the selected parameter values.

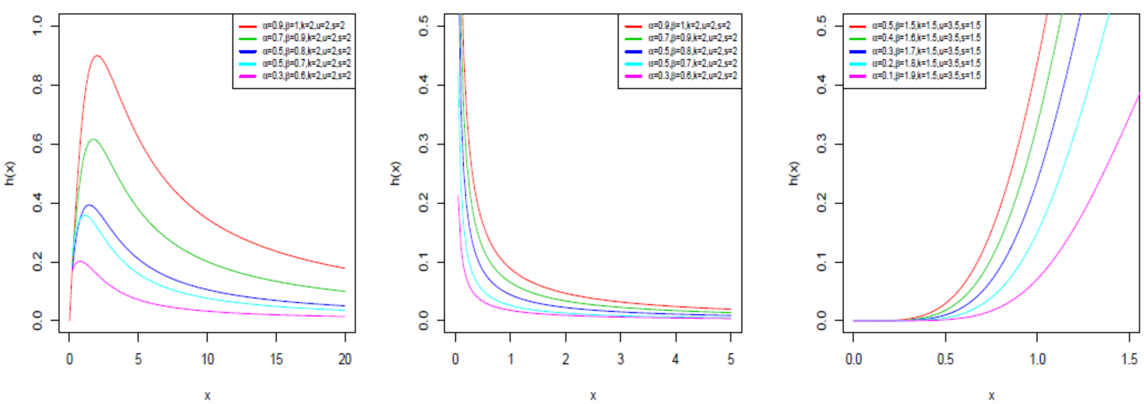

Figure 2: Plots of hrf of the GWBXII distribution for the selected parameter values

3. Properties of the Weibull-Burr XII Distribution

In this section, we study relation between GWBXII and Weibull, Exponential and Gumbel distribution, specifcally quantile function and shannon entropy.

Lemma 0.0.1 a) If a random variable Y follows the Weibull with parameters α and β distribution then the random variable X = $s(e^{\frac{Y}{k}} - 1)^{\frac{1}{u}}$ follows GWBXII(α, β, s, k, u)

b) If a random variable Y follows the standard exponential distribution, then the random variable X = $s((e^{\frac{-Y}{\alpha\beta}})^{\frac{-1}{k}} - 1)^{\frac{1}{\beta}}$ β follows GWBXII(α, β, s, k, u).

c) If a random variable Y follows the Type 1 extreme value distribution with scale parameter 1 , then the random variable X = $s((e^{\frac{-e^Y}{\alpha}})^{\frac{-1}{k}} - 1)^{\frac{1}{u}}$ follows GWBXII(c, γ, s, k, u).

The survival function (sf) (S(x)), hrf (h(x)) of GWBXII distribution are respectively given by

$$S(x;\alpha,\beta,s,k,u) = \exp\{-\alpha(-\log[1+(\frac{x}{s})^u]^{-k})^{\beta}.$$

$$h(x;\alpha,\beta,s,k,u) = \frac{\alpha\beta uks^{-u}x^{u-1}}{[1+(\frac{x}{s})^u]}\{-\log[1+(\frac{x}{s})^u]^{-k}\}^{\beta-1}$$

Lemma 0.0.2 Let Q(λ), 0 < λ < 1 denote the quantile function for the GWBXII. Then, Q(λ)is given by

$$Q(\lambda) = s((\mathrm{e}^{\frac{1}{\alpha}\ln(1-\lambda)})^{\frac{-1}{k}} - 1)^{\frac{1}{u}} \tag{5}$$

Setting λ= 0.25, 0.50, and 0.75 in (5), the quantiles of the GWBXII can be obtained.
Shannon′ s entropy for a random variable X with pdf f(x) is defined as E{− log f (x)}.

Lemma 0.0.3 The shannon′ s entropy for a random variable X that follows the GWBXII is

$$\eta = -\log(uks^{-u}) + (u-1)\frac{\gamma+\psi(k)}{u} + \frac{k+1}{k} - \frac{\Gamma(1+\frac{1}{\alpha})}{\beta\alpha} - \log(\alpha\beta)$$
$$+ (1-\beta)(\gamma - \frac{1}{\alpha}\log\beta + \frac{\alpha\Gamma(2-\frac{1}{\beta})}{\alpha^{2-\frac{1}{\beta}}}$$

Where $\psi = \frac{\Gamma'}{\Gamma}$.

Proposition 0.0.1 The Shannon entropy of a random variable X is defined by E{− log[f (x)]}.

Alzaatreh et al (2013) obtained, If a random variable X follows the family of distributions (3)

$$\eta_X = -E\{\log g(G^{-1}(1-\mathrm{e}^{-\mathrm{T}}))\} - \mu_{\mathrm{T}} + \eta_{\mathrm{T}},$$

where $\mu_T = \frac{\Gamma(1+\frac{1}{\alpha})}{\beta^{\alpha}}$ and $\eta_T = \gamma - \frac{1}{\alpha} log\ \beta$ are the mean and the Shannon entropy for the Weibull distribution and

$$-E\{\log g(G^{-1}(1-\mathrm{e}^{-\mathrm{T}}))\} = -E(\log g(x))$$
$$-E(\log(uks^{-u}x^{u-1}[1+(\frac{x}{s})^{u}]^{-k-1}) =$$
$$-\log(uks^{-u}) - (u-1)E(\log x) + (k+1)E(\log(1+(\frac{x}{s})^{u})] =$$
$$-\log(uks^{-u}) + (u-1)\frac{\gamma+\psi(k)}{u} + (k+1) + \frac{1}{k}$$

Hence

$$\eta = -\log(uks^{-u}) + (u-1)\frac{\gamma+\psi(k)}{u} + \frac{k+1}{k} - \frac{\Gamma(1+\frac{1}{\alpha})}{\beta^{\alpha}} - \log(\alpha\beta)$$
$$+ (1-\beta)(\gamma - \frac{1}{\alpha}\log\beta + \frac{\alpha\Gamma(2-\frac{1}{\beta})}{\alpha^{2-\frac{1}{\beta}}}$$

4. Skewness and Kurtosis Measures

The effects of the shape parameters on the skewness and kurtosis can be based on quantile measures. We obtain skewness and kurtosis measures using the qf. The Bowley's skewness measure is given by

$$\text{Skewness} = \frac{Q\left(\frac{1}{4}\right)+Q\left(\frac{3}{4}\right)-2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right)-Q\left(\frac{1}{4}\right)}$$

and the Moors's kurtosis measure is

$$\text{Kurtosis} = \frac{Q\left(\frac{7}{8}\right)-Q\left(\frac{5}{8}\right)+Q\left(\frac{3}{8}\right)-Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}$$

These measures enjoy the advantage of having less sensitivity to outliers.Moreover, they do exist for distribution without moments. Both measures equal zero for the normal distribution. Plots of skewness and kurtosis of the GWBXII distribution are presented in Figure 3. These plots indicate that both measures depend very much on the shape parameters and the proposed distribution can model various data types in terms of skewness and kurtosis.

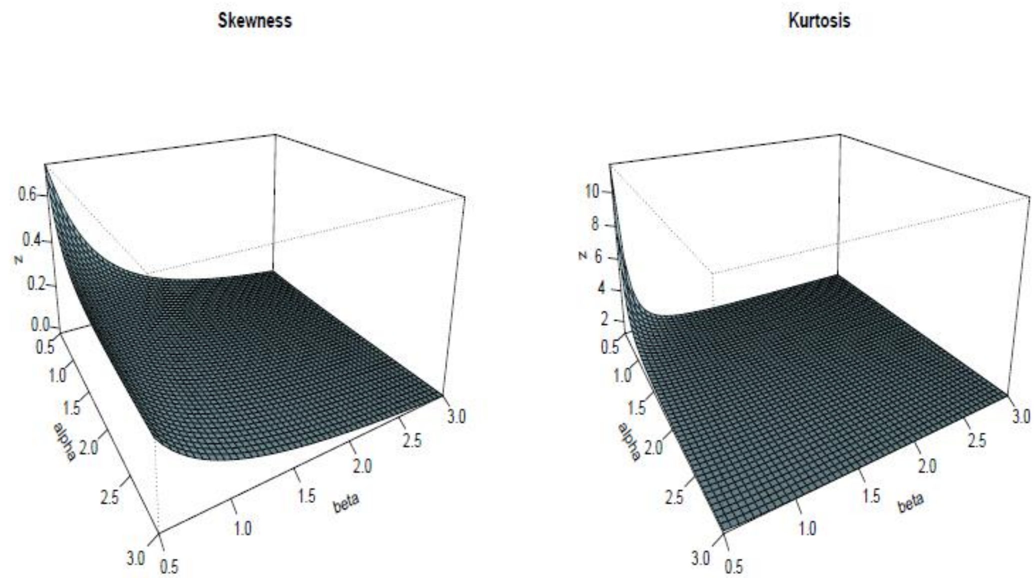

Figure 3:Plots of skewness and kurtosis of GWBXII distribution for k=2,u=2,s=2 parameters.

5. Useful expansions

For any real parameter c and $z\epsilon(0,1)$ it can be proven that, Cordeiro et al(2015) defined,

$$[-\log(1-z)]^c = z^c + c\sum_{i=0}^{\infty} p_i(c+i) z^{i+c+1} \tag{6}$$

where $p_i(c)$ are Stirling polynomials. The first six polynomials are $p_0(w) = 1/2$, $p_1(w) = (2 + 3w)/24$, $p_2(w) = (w + w2)/48$, $p_3(w) = (-8 - 10w + 15w^2 + 15w3)/5760$, $p_4(w) = (-6w - 7w^2 + 2w^3 + 3w^4)/11520$ and $p^5(w) = (96 + 140w - 224w^2 - 315w^2 + 63w^5)/2903040$.

These coefficients are related to the Stirling polynomials by $p_{n-1}(w) = S_n(w)/[n!(w + 1)]$ for $n \geq 1$ where $S_0(w) = 1$, $S_1(w) = (w + 1)/2$, etc.

By expanding the exponential function in 4, we can write

$$F(x) = \sum_{m=0}^{\infty} \frac{(-1)^{m+2}\alpha^{m+1}}{(m+1)!}\{-\log[1+(\frac{x}{s})^u]^{-k}\}^{(m+1)\beta}$$

and then using 6

$$F(x)=\sum_{m=0}^{\infty}\frac{(-1)^{(m+2)}\alpha^{m+1}}{(m+1)!}\{(1-(1+(\frac{x}{s})^u)^{-k})^{(m+1)\beta}$$
$$+(m+1)\beta\sum_{i=0}^{\infty}p_i[(m+1)\beta+i](1-(1+(\frac{x}{s})^u)^{-k})^{i+(m+1)\beta+1}\}$$

Expanding $(1-(1+(x/s)^u)^{-k})[(m+1)\beta+i]$ and $(1-(1+(x/s)^u)^{-k})^{i+(m+1)\beta+1}$ in power series, F(x) can be expressed as

$$F(x)=\sum_{a=0}^{\infty}w_a(1-(1+(\frac{x}{s})^u)^{-k})^a=\sum_{a=0}^{\infty}w_aH_a(x) \tag{7}$$

where $H_a(x)$ denotes the cdf of the Exp-Burr XII(a) distribution and

$$w_a=\sum_{m,j=0}^{\infty}\frac{(-1)^{m+j+a+2}\alpha^{m+1}}{(m+1)!}\binom{j}{a}\{\binom{(m+1)\beta}{j} \tag{8}$$
$$+(m+1)\beta\sum_{i=0}^{\infty}p_i[(m+1)\beta+i]\times\binom{i+(m+1)\beta+1}{j}\} \tag{9}$$

The corresponding pdf of X can be expressed as

$$f(x)=\sum_{a=0}^{\infty}(a+1)\nu_a(1-(1+(\frac{x}{c})^u)^{-k})^a uks^{-u}x^{u-1}[1+(x/s)^u]^{-k-1}=\sum_{a=0}^{\infty}\nu_a h_{a+1}(x) \tag{10}$$

where $h_{a+1}(x)$ denotes the pdf of the Exp-Burr XII(a + 1) distribution and $V_a = w_{a+1}$.

6. Moments

Some of the most important features and characteristics of a distribution can be studied through moments (e.g. tendency, dispersion, skewness and kurtosis).

The nth moment of the GWBXII distribution is given by

$$\mu_n'=\sum_{a=0}^{\infty}\nu_a s^n k(a+1)B[k(a+1)-nu^{-1},1+nu^{-1}],\quad k(a+1)-nu^{-1}>0$$

Table 1 represents the first four raw moments of GWBXII for some selected parameter vectors. Based on the Table 1, it is clear that when α increases, mean is decreases, similarly, when β increases, mean is decreases.

Table 1: Moments of the GWBXII distribution

Parameters	μ_1'	μ_2'	μ_3'	μ_4'
(0.5,2,2,2,2)	1.855	3.928	9.288	24.208
(1,2,2,2,2)	1.467	2.408	4.329	8.418
(1.5,2,2,2,2)	1.291	1.851	2.886	4.832
(1.5,1.5,2,0.5,0.5)	0.181	0.193	0.818	13.221
(1.5,2,2,0.5,0.5)	0.151	0.065	0.056	0.0863
(1.5,2.5,2,0.5,0.5)	0.144	0.0432	0.021	0.015

7. Generating function

Here, we provide two formulae for the moment generating function (mgf) M (t) = E(e^{tx}) of X. A first formula for M(t) comes from 10 as

$$M(t) = \sum_{a=0}^{\infty} \nu_a M_{a+1}(t) \tag{11}$$

where $M_{a+1}(t)$ is the mgf of the BXII(s, k(a+1), u) distribution. We provide a simple representation for the mgf M_{BXII} (t) of the BXII(s, k, u) distribution.We can write for t < 0

$$M_{BXII}(t) = uk \int_0^{\infty} \exp(xst) x^{u-1} (1 + x^u)^{-(k+1)} dx.$$

Now, we need the Meijer G-function defined by

$$G_{p,q}^{m,n}\left(x \,\middle|\, \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \right) = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m} \Gamma(b_j + t) \prod_{j=1}^{n} \Gamma(1 - a_j - t)}{\prod_{j=n+1}^{p} \Gamma(a_j + t) \prod_{j=m+1}^{p} \Gamma(1 - b_j - t)} x^{-1} \mathrm{d}t.$$

where i =$\sqrt{-1}$ is the complex unit and L denotes an integration path; see Section 9.3 in Gradshteyn and Ryzhik (2000) for a description of this path. The- Meijer G-function contains as particular casesmany integrals with elementary and special functions (Prudnikov et al., 1986). We now assume that u= m/k, where m and k are positive integers. This condition is not restrictive since every positive real number can be approximated by a rational number. Using the integral (12) given in Appendix A, we conclude for t < 0

$$M_{BXII}(t) = mI'(-st, mk^{-1} - 1, mk^{-1}, -k - 1)$$

Hence, from Eq. 11, the mgf of the GWBXII(α, β, s, k, u) distribution for t < 0 follows as

$$M(t) = m \sum_{a=0}^{\infty} \nu_a I'(-st, \frac{m}{k(a+1)} - 1, \frac{m}{k(a+1)}, -k(a+1) - 1)$$

8. Maximum Likelihood Estimation

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the new family of distributions from complete samples only. Let $X_1, ..., X_n$ be a sample of size n from the GWBXII distribution with parameters α,β, s,k,u. The log-likelihood function for the vector of parameters Θ=(α,β, s,k,u) can be expressed

$$\ell(\Theta) = n\log(\alpha\beta uks^{-u}) - \Sigma_{i=1}^{n}\log(1+(\frac{x}{s})^u) + (\beta-1)\sum_{i=1}^{n}\log(-\log(1+(\frac{x}{s})^u)^{-k})$$
$$+\sum_{i=1}^{n}\alpha(-\log[1+(\frac{x}{s})^u]^{-k})^\beta$$

The components of the score vector D(Θ) = (D_α, D_β, D_s, D_k, D_u) are

$$D_\alpha = \frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n}\{-\log[1+(\frac{x_i}{s})^u]^{-k})\}^\beta$$

$$D_\beta = \frac{\partial\ell}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log(-\log[1+(\frac{x_i}{s})^u]^{-k}) - \sum_{i=1}^{n}\alpha(-\log[1+(\frac{x_i}{s})^u]^{-k})^\beta$$
$$\log(-\log[1+(\frac{x_i}{s})^u]^{-k}),$$

$$D_u = \frac{\partial\ell}{\partial u} = \frac{n}{u} - n\log s - \sum_{i=1}^{n}\frac{(\frac{x_i}{s})^u\log\frac{x_i}{s}}{1+(\frac{x_i}{s})^u}$$
$$-(\beta-1)\sum_{i=1}^{n}k[1+(\frac{x_i}{s})^u]^{-1}(\frac{x_i}{s})^u\log\frac{x_i}{s}\log[1+(\frac{x_i}{s})^u]^k$$
$$+\Sigma_{i=1}^{n}\alpha\beta(-\log[1+(\frac{x_i}{s})^u]^{-k})^{\beta-1}k[1+(\frac{x_i}{s})^u]^{-1}(\frac{x_i}{s})^u\log\frac{x_i}{s},$$

$$D_k = \frac{\partial\ell}{\partial k} = \frac{n}{k} - (\beta-1)\sum_{i=1}^{n}\log[1+(\frac{x_i}{s})^u]^{k+1} + \sum_{i=1}^{n}\beta\alpha(-\log[1+(\frac{x_i}{s})^u]^{-k})^{\beta-1}$$
$$\log[1+(\frac{x_i}{s})^u],$$

$$D_s = \frac{\partial\ell}{\partial s} = \frac{-nu}{s} + (\beta-1)\sum_{i=1}^{n}k[1+(\frac{x_i}{s})^u]^{-1}x_i^u\frac{us^{u-1}}{s^{2u}}\log[1+(\frac{x_i}{s})^u]^k$$
$$+\sum_{i=1}^{n}\beta\alpha(-\log[1+(\frac{x_i}{s})^u]^{-k})^{\beta-1}k[1+(\frac{x_i}{s})^u]^{-1}x_i^u\frac{us^{u-1}}{s^{2u}} - \sum_{i=1}^{n}\frac{u(\frac{x_i}{s})^{u-1}\frac{-x_i}{s^2}}{1+(\frac{x_i}{s})^u}.$$

Setting Dα; Dβ ; Ds; Dk; Duequal to zero and solving the equations simul- taneously yields the maximum likelihood estimates $\hat{\Theta} = (\hat{\alpha}, \hat{\beta}, \hat{s}, \hat{k}, \hat{u})$ of Θ = (α, β, s, k, u).

9. Simulation Study

In this section, a simulation study is conducted to examine the performance of the MLEs of GWBXII parameters. Inverse transform method is used to generate random observations from GWBXII distribution. We generate 1000 samples of size, n =50, 100, 500 and n=1000 of GWBXII distribution. The evaluation of estimates was based on the bias of the MLEs of the model parameters, the mean squared error (MSE) of the MLEs. The empirical study was conducted with soft- ware R and the results are given in Table 2. The values in Table 2 indicate that the estimates are quite stable and, more importantly, are close to nominal values when n goes to infinity. It is observed from Table 2 that the biases and MSEs decreases as n increases. The simulation study shows that the maximum likeli- hood method is appropriate for estimating the GWBXII parameters. In fact, the MSEs of the parameters tend to be closer to the zero when n increases. This fact supports that the asymptotic normal distribution provides an adequate approxi- mation to the finite sample distribution of the MLEs. The normal approximation can be improved by using bias adjustments to these estimators.

Table 2: Biases and MSEs for the MLEs of the parameters of the W-BXII distribution.

k=0.5,u=2,s=0.5			Bias					MSE				
α	β	n	α	β	k	u	s	α	β	k	u	s
0.5	2	50	0.141	0.158	0.047	0.274	0.617	0.016	0.167	0.037	0.406	0.097
		100	0.091	0.096	0.040	0.109	0.083	0.011	0.161	0.031	0.230	0.073
		500	0.063	0.037	0.009	0.020	0.018	0.002	0.030	0.006	0.046	0.012
		1000	0.007	0.005	0.004	0.022	0.002	0.001	0.026	0.004	0.029	0.006
2	2	50	0.135	0.138	0.143	0.057	0.120	0.131	0.154	0.199	0.324	0.155
		100	0.102	0.079	0.08	0.045	0.068	0.004	0.064	0.105	0.164	0.075
		500	0.022	0.012	0.014	0.017	0.014	0.0002	0.019	0.011	0.031	0.008
		1000	0.001	0.009	0.006	0.011	0.006	0.0001	0.015	0.006	0.021	0.005
5	3	50	0.107	0.119	0.117	0.048	0.086	0.0245	0.0125	0.131	0.121	0.071
		100	0.049	0.031	0.081	0.003	0.059	0.0112	0.007	0.072	0.075	0.038
		500	0.012	0.014	0.023	0.0009	0.017	0.056	0.003	0.0137	0.023	0.007
		1000	0.004	0.005	0.011	0.002	0.007	0.003	0.001	0.0036	0.008	0.002

10. Applications

In this section, we compare the fiGing performance of GWBXII model with BXII,BBXII (Paranaba et al.(2011)) ,KWBXII (Paranaba et al.(2013)) ,MCW (Cordeiro et al (2014)) and BW (Famoye et al. (2005)) models with following densities:

$$f_{BBXII}(x;a,b,s,k,u) = \frac{ukx^{u-1}}{s^u B(a,b)}(1+(x/s)^u)^{-(kb+1)}(1-[1+(x/s)^u]^{-k})^{a-1}$$

$$f_{KWBXII}(x;a,b,s,k,u) = abuks^{-u}x^{u-1}(1+(x/s)^u)^{-k-1}(1-[1+(x/s)^u]^{-k})^{a-1}$$
$$(1-[1-\{1+(x/s)^u\}^{-k}]^a)^{b-1}$$

$$f_{MCW}(x;a,b,w,\alpha,\beta) = \frac{w\alpha\beta^\alpha}{B(a/w,b)}x^{\beta-1}\exp(-[\beta x]^\alpha)(1-\exp[-(\beta x)^\alpha])^{a-1}$$
$$\{1-[1-\exp[-(\gamma x)^c]^w\}^{b-1}$$

$$f_{BW}(x;a,b,\alpha,\beta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1}(1-\mathrm{e}^{-(\frac{x}{\gamma})^{\alpha-1}})^{\mathrm{a}-1}\mathrm{e}^{-\mathrm{b}(\frac{x}{\beta})^\alpha}$$

where a, b, s, k, u > 0 and B(a,b) is the beta function.

The data set is taken from Gross and Clark (1975) and represents the relief times of 20 patients receiving analgesic. The data set:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2,1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6,2.0

Table 3 gives W* and A* statistics and estimated parameters. Based on Tables 3, it is clear that GWBXII distribution provides the overall best fit and therefore could be chosen as the more adequate model from other models for explaining the used data set. More information can be provided in Figure 4 by a histogram of the data with fiGed lines of the pdfs for all distributions. Figure 4 suggests that the GWBXII fits left-skewed data very well.

Table 1: The MLEs(standard errors in parentheses), A^* and W^*

Model	Estimation	W*	A*
$GWBXII(\alpha, \beta, s, k, u)$	0.16 , 0.88 , 1.55 , 3.67 , 9.88 (0.07) , (0.15) , (0.004) , (0.005) , (0.005)	0.02	0.12
$BBXII(a, b, s, k, u)$	8.42 , 0.16 , 2.47 , 19.99 , 2.78 (14.19) , (0.05) , (0.33) , (0.33) , (0.67)	0.04	0.27
$BXII(s, k, u)$	1.45 , 0.38 , 9.59 (0.17) , (0.25) , (3.92)	0.022	0 .27
$KWBXII(a, b, s, k, u)$	0.79 , 0.126 , 1.44 , 2.61 , 11.05 (0.05) , (0.03) , (0.06) , (0.08) , (0.05)	0.02	0.13
$MCW(a, b, w, \alpha, \beta)$	1.94 , 0.24 , 19.17 , 3.09 , 0.26 (1.66) , (0.24) , (14.93) , (1.36) , (0.34)	0.08	0.49
$BW(a, b, \alpha, \beta)$	8.19 , 0.19 , 1.85 , 1.97 (4.34) , (0.06) , (0.10) , (0.09)	0.08	0.46

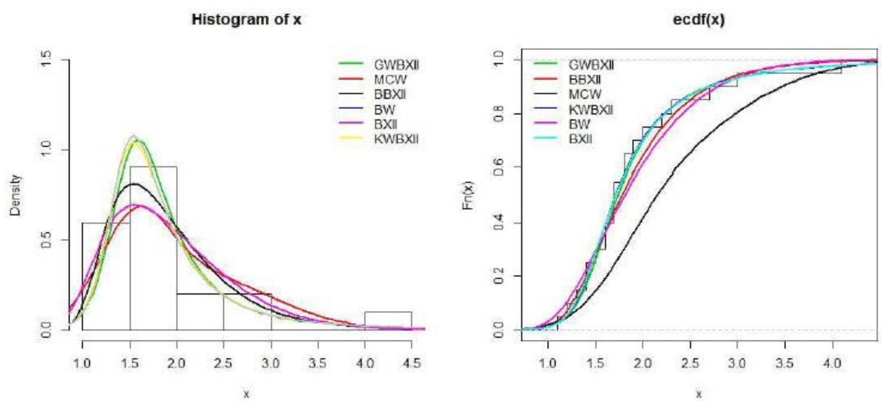

Figure 4: Estimated pdf and cdf for the GWBXII,BBXII, BXII, KWBXII,MCW and BW model.

11. Summary

A new lifetime distribution is introduced and provided some of its mathe- matical properties including the quantile function, entropy and moments. The maximum likelihood method is used to estimated the model parameters. For different parameter seGings and sample sizes, a simulation study is performed to evaluate the performance of the MLEs of GWBXII parameters. Empirical results show that the new model provides a consistently beGer fit than other extended Burr XII models such as the BBXII and KWBXII distributions. Thus, we hope that the new model will be useful for practitioners.

Appendix A

We have the following result which holds for m and k positive integers, $\mu > -1$ and $p > 0$ (Prudnikov et al., 1992, page 10 21):

$$I'(p,\mu,\frac{m}{k},\nu) = \int_0^\infty \exp(-px)x^\mu(1+x^{\frac{m}{k}})^\nu dx$$
$$= \frac{k^{-\nu}m^{\mu+\frac{1}{2}}}{(2\pi)^{\frac{(m-1)}{2}}\Gamma(-\nu)p^{\mu+1}} G^{k,k+m}_{k+m,k}\left(\frac{m^m}{p^m} \middle| \begin{array}{c} \Delta(m,-\mu),\Delta(k,\nu+1) \\ \Delta(k,0), \end{array} \right) \quad (12)$$

where $\Delta(k, a) = \frac{a}{k}, \frac{a+1}{k}, \ldots, \frac{a+k}{k}$.

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