

On the extension of inverse Lindley distribution

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Abstract

In this paper, we proposed another extension of inverse Lindley distribution, called extended inverse Lindley and studied its fundamental properties such as moments, inverse moments, mean deviation, stochastic ordering and entropy. The flexibility of the proposed distribution is shown by studying monotonicity properties of density and hazard functions. It is shown that the distribution belongs to the family of upside-down bathtub shaped distributions. Maximum likelihood estimators along with asymptotic confidence intervals are constructed for estimating the unknown parameters. An algorithm is presented for random number generation from the distribution. The property of consistency of MLEs has been verified on the basis of simulated samples. The applicability of the extended inverse Lindley distribution is illustrated by means of real data analysis.

Keywords: Extended inverse Lindley distribution, upside-down bathtub shaped hazard rate, Moments, Maximum likelihood estimation.

1. Introduction

In 1958, Prof. D.V. Lindley (Lindley (1958)) investigated a probability distribution in con-text of fiducial statistic as a counter example of Bayesian theory. Later, this distribution is called as the Lindley distribution (LD). Ghitany et al. (2008) discussed the fundamental properties of the LD with application to waiting time data. Mazucheli and Achcar (2011) worked on the Lind-ley distribution applied to competing risks lifetime data. Krishna and Kumar (2011) estimated the parameter of Lindley distribution with progressive Type-II censoring scheme. They also showed that it may be better lifetime model than exponential, lognormal and gamma distribu-tions in some real life situations. Since then the distribution has been widely discussed in various context. Singh and Gupta (2012) have used the Lindley distribution under load sharing sys-tem models. Al-Mutairi et al. (2013) developed the inferential procedure of the stress-strength parameter, when both stress and strength variables follow Lindley distribution.

It may be mentioned here that the Lindley distribution is useful when the data show increas-ing failure rate. This is the property that encourage the use of Lindley distribution in lifetime data analysis over exponential distribution. Although the family of Lindley distributions posses very nice properties and gained great applicability in various disciplines, its applicability may

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be restricted to non-monotone hazard rate data (bathtub and upside down bathtub (UBT) see Sharma et al. (2014a)). Therefore, the LD has been extended to various ageing classes and introduced various generalized class of lifetime distribution based on Lindley distribution. Zakerzadeh and Dolati (2009) introduced three parameters extension of the Lindley distribution. Nadarajah et al. (2011), Ghitany et al. (2013) proposed two parameter generalizations of the Lindley distribution, called as the generalized Lindley and power Lindley distributions. These distributions are generated using the exponentiation and power transformations to the Lindley distribution. Merovci (2013) and Merovci and Elbatal (2014) investigated transmuted Lindley and transmuted Lindley-geometric distributions respectively. The beta-Lindley distribution is introduced by Merovci and Sharma (2014). Statistical and mathematical properties of Kumaraswamy Quasi Lindley and Kumaraswamy Lindley distributions are discussed by Elbatal and Elgarhy (2013) and Akmakyapan and Kadlar (2014) respectively. The exponentiated power Lindley distribution is introduced by Ashour and Eltehiwy (2015). The generalized Poisson-Lindley and another extension of the Lindley distribution are discussed by Mahmoudi and Zakerzadeh (2010) and Oluyede and Yang (2015).

In the references cited above, authors mainly focused on the estimation of increasing, decreasing and bathtub shaped failure rates data. Nobody has paid attention to the modelling of the upside down bathtub data. Recently, Sharma et al. (2015) investigated inverted version of the Lindley distribution that has UBT shaped failure rate. The inverse Lindley distribution (ILD) is defined by the following probability density function (pdf)

$$f(x; \theta) = \left(\frac{\theta^2}{1 + \theta} \right) \left(\frac{1 + x}{x^3} \right) e^{-\theta/x}, x > 0, \theta > 0. \quad (1)$$

The above density (1) can be rewrite as

$$f(x) = pf_1(x) + (1 - p)f_2(x), \quad (2)$$

where, $p = \frac{\theta}{1 + \theta}$, $f_1(x) = \theta x^{-2} e^{-\theta/x}$, $x > 0, \theta > 0$ and $f_2(x) = \frac{\theta^2}{\Gamma^2} x^{-3} e^{-\theta/x}$, $x > 0, \theta > 0$.

Thus, inverse Lindley distribution is a two component mixture of inverse exponential distribution and special case of inverse gamma distribution. Sharma et al. (2015) discussed the properties of inverse Lindley distribution with application to stress strength reliability analysis. Sharma et al. (2014b) introduced two parameter extension of inverse Lindley distribution using power transformation to inverse Lindley random variable. Recently, Alkarni (2015) proposed three parameter inverse Lindley distribution with application to maximum flood level data.

In this article, we proposed another extension of inverse Lindley distribution which offers more flexibility with a effective shape parameter. The pdf of the extension is defined by

$$f(x; \alpha, \theta) = p \text{InverseGamma}(\alpha - 1, \theta) + (1 - p) \text{InverseGamma}(\alpha, \theta), \alpha > 1, \theta > 0, \quad (3)$$

where $p = \frac{\theta}{1 + \theta}$, and $\text{InverseGamma}(\alpha, \theta)$ stands for inverse gamma distribution with shape parameter α and scale parameter θ i.e. $f(x) = \frac{\theta^\alpha}{\Gamma(\alpha)} \frac{e^{-\frac{\theta}{x}}}{x^{\alpha+1}}$, $x > 0, \theta > 0, \alpha > 0$. The pdf of the EILD is given by

$$f(x; \alpha, \theta) = \frac{\theta^\alpha}{(1 + \theta)\Gamma(\alpha)} \frac{(1 + x(\alpha - 1))}{x^{\alpha+1}} e^{-\theta/x}, x > 0, \theta > 0, \alpha > 1. \quad (4)$$

The corresponding cumulative distribution function of (3) is given by

$$F(x; \alpha, \theta) = p \text{CDFInverseGamma}(\alpha - 1, \theta) + (1 - p) \text{CDFInverseGamma}(\alpha, \theta), \alpha > 1, \theta > 0, \quad (5)$$

where, $CDFInverseGamma(\alpha, \theta) = \frac{\Gamma(\theta/x, \alpha)}{\Gamma\alpha}$, $\Gamma(t, \alpha) = \int_0^t x^{\alpha-1} e^{-x} dx$ and $\Gamma(\alpha) = \int_0^\infty t^{(\alpha-1)} e^{-t} dt$.

We call the distribution in (3) as the extended inverse Lindley distribution and denote the density as $EILD(\alpha, \theta)$. Note that if X follows $EILD(\alpha, \theta)$, then $Y = X^{-1}$ follows the generalized Lindley distribution by Abouammoha et al. (2015).

Rest of the article is organized in the following sections. The distributional properties such as moments, skewness, kurtosis, mean deviation, stochastic ordering and Reyni entropy are investigated in section 1. The maximum likelihood estimation along with asymptotic distribution is discussed in section 2. Section 3 consists algorithm for random sample generation form the EILD. Simulation study has also been carried out study the performances of the MLEs. A set of real data is used for illustration purposes in section 4. The paper is concluded in section 5.

2. Distributional properties

In this section, we discuss the fundamental properties of the EILD. Mainly the properties to be discussed include measures of central tendency, dispersion and shapes of the frequency distribution from EILD. First, we study the shapes of the pdf and hazard function in the following section.

2.1. Shapes of Density and Hazard functions

The pdf of extended inverse Lindley distribution is given by

$$f(x; \alpha, \theta) = \frac{\theta^\alpha (1 + x(\alpha - 1))}{(1 + \theta)\Gamma(\alpha) x^{\alpha+1}} e^{-\theta/x}, x > 0, \theta > 0, \alpha > 1.$$

The first derivative of the pdf with respect to x is readily obtained as

$$f'(x) = \frac{\theta^\alpha (\theta - (\alpha - \theta + 1)x - \alpha x^2)}{(\theta + 1)\Gamma(\alpha) x^{\alpha+3}} e^{-\theta/x}.$$

By putting $f'(x) = 0$, we get the mode of EILD which is given by

$$x_{mode} = \frac{-(\alpha\theta - \theta + \alpha + 1) \pm \sqrt{(\alpha\theta - \theta + \alpha + 1)^2 + 4(\alpha\theta - \alpha^2\theta)}}{2(\alpha - \alpha^2)}.$$

For various shapes of the pdf of EILD, see Figure 1a. The survival function of the density is given by $S(x) = 1 - F(x)$. The survival function of EILD have been plotted in Figure 1b for various combination of α and θ .

The hazard function of extended inverse Lindley distribution is given by

$$h(x) = \frac{\theta^\alpha \Gamma(\alpha) e^{-(\theta/x)} (1 + x(\alpha - 1))}{x^{\alpha-1} [(\theta + 1)\Gamma\alpha - \{\gamma(\alpha, \theta/x) + \theta(\alpha - 1)\gamma(\alpha - 1, \theta/x)\}]} \tag{6}$$

The hazard function of EILD for various choices of α and θ is shown in Figure 2 indicates that the hazard function is upside down bathtub shaped.

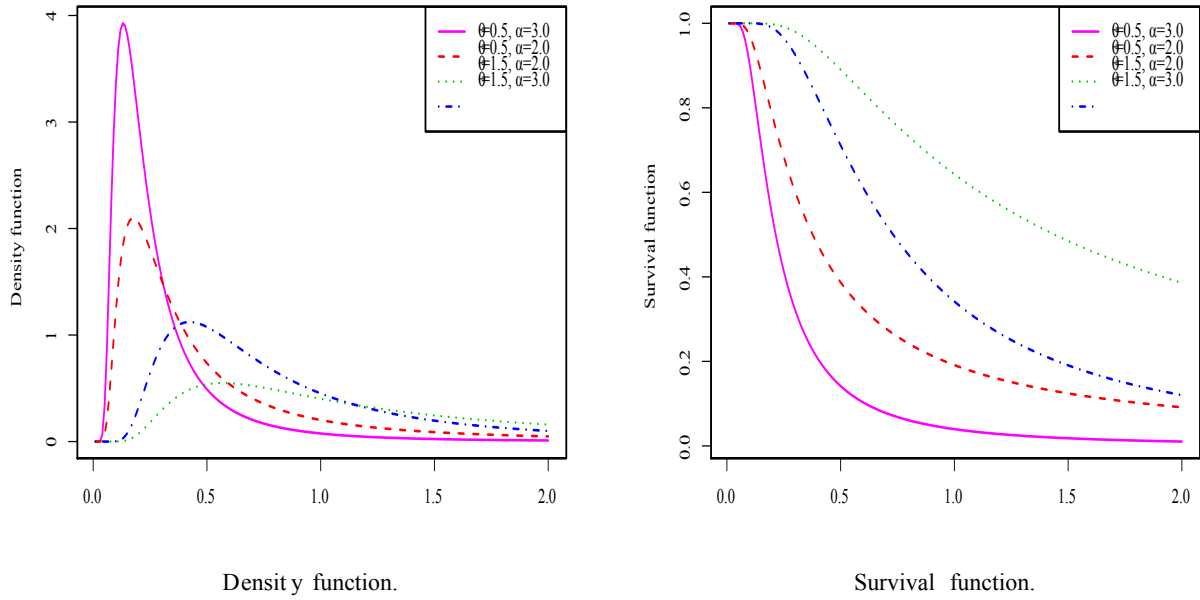


Figure 1: Density and survival functions of EILD(α, θ).

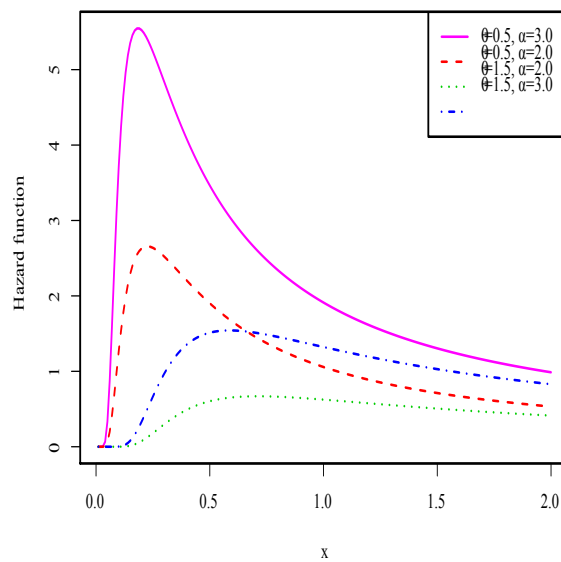


Figure 2: The hazard function of the EILD(α, θ).

2.2. Moments, inverse moments and associated Measures

The r^{th} moment can be defined by,

$$E [X^r] = \int_0^{\infty} x^r f(x; \alpha, \theta) dx.$$

On putting the density function, we get

$$E [X^r] = \int_0^{\infty} x^r \frac{\theta^\alpha}{(1 + \theta)\Gamma(\alpha)} \frac{(1 + x(\alpha - 1))}{x^{\alpha+1}} e^{-\theta/x} dx.$$

Using the definition of inverse gamma, we obtain

$$E[X^r] = \frac{\theta^r [(\theta + 1)(\alpha - 1) - r] \Gamma(\alpha - r - 1)(\theta + 1)}{\Gamma(\alpha)}$$

The r^{th} inverse moment $E \left[\frac{1}{X^r} \right]$ can be obtained by

$$E \left[\frac{1}{X^r} \right] = \int_0^{\infty} \frac{1}{x^r} f_X(x) dx.$$

After substituting the pdf of the ILD, we have

$$E \left[\frac{1}{X^r} \right] = \int_0^{\infty} \frac{1}{x^r} \frac{\theta^\alpha}{(1 + \theta)\Gamma(\alpha)} \frac{(1 + x(\alpha - 1))}{x^{\alpha+1}} e^{-\theta/x} dx.$$

On some simplifications, we get

$$E \left[\frac{1}{X^r} \right] = \frac{[(\alpha - 1)(1 + \theta) + r] \Gamma(\alpha + r - 1)(\theta + 1)}{+ 1)\theta^r \Gamma(\alpha)}.$$

The harmonic mean of the EILD can be computed using the first inverse moment as $1/E \left[\frac{1}{X} \right]$. The first four moments of EILD are given by

$$\mu'_1 = E[x] = \frac{\theta[(\theta + 1)(\alpha - 1) - 1]}{(\theta + 1)(\alpha - 1)(\alpha - 2)}, \alpha > 2,$$

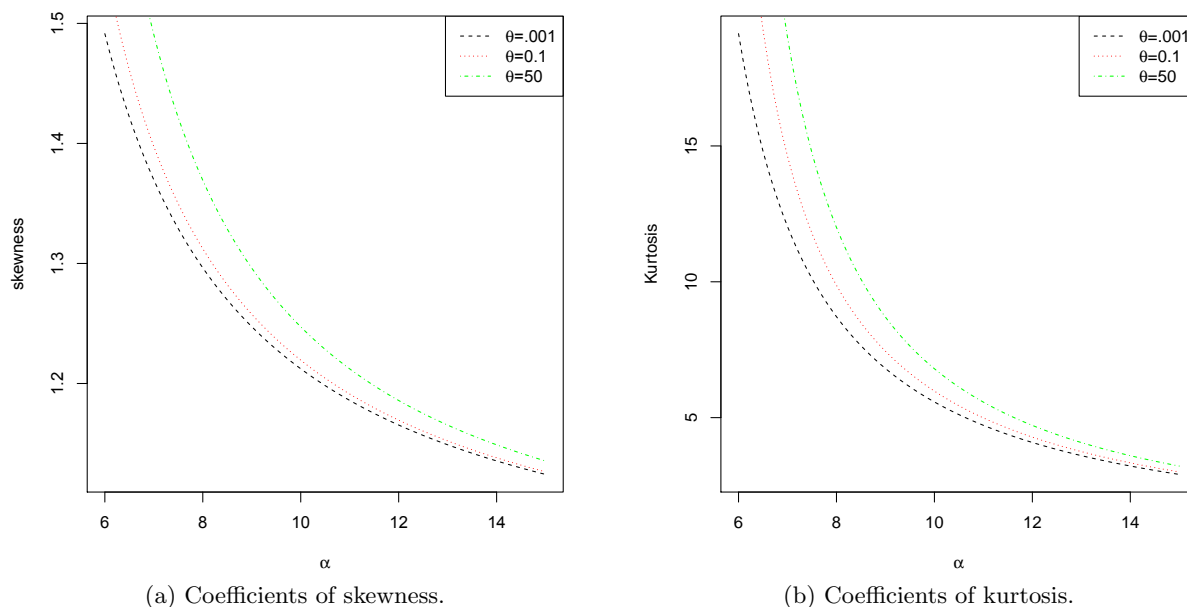
$$\mu'_2 = \frac{\theta^2[(\theta + 1)(\alpha - 1) - 2]}{(\theta + 1)(\alpha - 1)(\alpha - 2)(\alpha - 3)}, \alpha > 3,$$

$$\mu'_3 = \frac{\theta^3[(\theta + 1)(\alpha - 1) - 3]}{(\theta + 1)(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)}, \alpha > 4,$$

$$\mu'_4 = \frac{\theta^4[(\theta + 1)(\alpha - 1) - 4]}{(\theta + 1)(\alpha - 1)(\alpha - 2)(\alpha - 3)(\alpha - 4)(\alpha - 5)}, \alpha > 5.$$

The variance of the EILD can be computed using the following relation,

$$\sigma^2 = \mu'_2 - \mu_1'^2.$$

Figure 3: Plots for skewness and kurtosis of EILD(α, θ).

The variance of the EILD random variable is given by

$$\sigma^2 = \frac{[(\theta + 1)^2(\alpha - 1)^2 - 2(\theta + 1)(\alpha - 1) - (\alpha - 3)]}{(\theta + 1)^2(\alpha - 1)^2(\alpha - 2)^2(\alpha - 3)}, \quad \alpha > 3.$$

The coefficients of the skewness and kurtosis measures can now be computed from the following expression,

$$\text{skewness}(\gamma_1) = \frac{[\mu_3' - 3\mu_1'\mu_2' + 2(\mu_1')^3]}{[\mu_2' - (\mu_1')^2]^{3/2}}, \quad \alpha > 3$$

$$\text{kurtosis}(\gamma_2) = \beta_2 - 3$$

where,

$$\beta_2 = \frac{[\mu_4' - 4\mu_1'\mu_3' + 6\mu_1'^2\mu_2' - 3\mu_1'^4]}{[\mu_2' - (\mu_1')^2]^2}, \quad \alpha > 4.$$

The skewness for EILD is shown in Figure 3a for various choices of α and θ . From the figure it can be observed that the $\gamma_1 > 0$ so the EILD is positively skewed. The kurtosis is shown in Figure 3b for various values of α and θ . It can be observed from the figure that $\gamma_2 > 0$. EILD is more peaked than normal curve so it is leptokurtic.

2.3. Mean Deviation and Median Deviation

Lemma 1. Let

$$I(x; A) = \int_A^\infty xf(x)dx$$

then

$$I(x; A) = \frac{\theta}{(1 + \theta)\Gamma\alpha} \left[\Gamma\left(\frac{\theta}{A}, \alpha - 1\right) + \theta(\alpha - 1)\Gamma\left(\frac{\theta}{A}, \alpha - 2\right) \right], \quad \alpha > 2.$$

Proof Consider the given integral,

$$I(x; A) = \int_A^\infty xf(x)dx = \frac{\theta^\alpha}{(1 + \theta) \Gamma\alpha} \int_A^\infty \frac{(1 + x(\alpha - 1))}{x^\alpha} e^{-\theta/x} dx.$$

On some simplifications, we get θ

$$I(x; A) = \frac{1}{(1 + \theta)\Gamma\alpha} \left[\Gamma\left(\frac{\theta}{A}, \alpha - 1\right) + \theta(\alpha - 1)\Gamma\left(\frac{\theta}{A}, \alpha - 2\right) \right].$$

The mean deviation about mean and median are defined by,

$$M_1(x) = \int_0^\infty (x - \mu)f_X(x)dx = M_1(x) = 2\mu F(\mu) + 2 \int_\mu^\infty xf_X(x)dx - \mu - 1,$$

and

$$M_2(x) = \int_0^\infty (x - m)f_X(x)dx = M_2(x) = 2 \int_m^\infty xf_X(x)dx - 1,$$

respectively, where μ and m represent the mean and median of the EILD. Using the Lemma 1, we get

$$M_1(x) = 2\mu F(\mu) + 2I(x; \mu) - \mu - 1$$

and

$$M_2(x) = 2I(x; m) - 1.$$

2.4. Stochastic ordering

A random variable X is said to be stochastically greater than Y if $F_X(x) \leq F_Y(x)$ for all x . In the similar way, X is said to be stochastically greater than Y in the

- hazard rate order ($Y \leq_{hr} X$) if $h_X(x) \leq h_Y(x)$ for all x .
- mean residual life order ($Y \leq_{mrl} X$) if $m_X(x) \leq m_Y(x)$ for all x .
- likelihood ratio order ($Y \leq_{lr} X$) if $\frac{f_X(x)}{f_Y(x)}$ is an increasing function of x .

Shaked and Shanthikumar (1994) have stated the results for stochastic ordering of distribution as follows:

$$X \leq_{lr} Y \implies X \leq_{hr} Y \implies X \leq_{mrl} Y$$

\downarrow
 $X \leq_{st} Y$

Following the above relations, to show the above ordering in hr, mrl and lr, it is sufficient to show the ordering in likelihood only.

Proposition 1: Let X and Y are two independent random variables follow the extended inverse Lindley distribution with shape parameters α_1 and α_2 and scale parameters θ_1 and θ_2 respectively. If $\theta_2 \geq \theta_1$, then ($Y \leq X$) and if $\alpha_1 \geq \alpha_2$, then ($Y \leq X$) for all x , while other parameters kept fixed and same.

Proof It is given that $Y \sim \text{EILD}(\alpha_1, \theta_1)$ and $X \sim \text{EILD}(\alpha_2, \theta_2)$. Then the likelihood ratio is given by,

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_2^{\alpha_2} e^{-\theta_2/x} (\theta_1 + 1) x^{\alpha_1 - 1} \{1 + x(\alpha_2 - 1)\}}{\theta_1^{\alpha_1} e^{-\theta_1/x} (\theta_2 + 1) x^{\alpha_2 - 1} \{1 + x(\alpha_1 - 1)\}}$$

Now for $\alpha_1 = \alpha_2 = \alpha$, we have

$$\frac{f_X(x)}{f_Y(x)} = \frac{(1 + \theta_1)}{(1 + \theta_2)} \left(\frac{\theta_2}{\theta_1}\right)^\alpha \exp\left[\frac{-(\theta_2 - \theta_1)}{x}\right],$$

and

$$\log\left(\frac{f_X(x)}{f_Y(x)}\right) = \log(1 + \theta_1) - \log(1 + \theta_2) + \alpha \log\left(\frac{\theta_2}{\theta_1}\right) - \frac{(\theta_2 - \theta_1)}{x}.$$

On differentiating, we get

$$\frac{\delta}{\delta x} \left\{ \log\left(\frac{f_X(x)}{f_Y(x)}\right) \right\} = \frac{(\theta_2 - \theta_1)}{x^2}.$$

We can easily find that the function above is increasing in x for $\theta_2 \geq \theta_1$. Which means that X is stochastically greater than Y with respect to likelihood ratio if $\theta_2 \geq \theta_1$ and $\alpha_1 = \alpha_2 = \alpha$.

If $\theta_1 = \theta_2 = \theta$, the likelihood ratio becomes,

$$\frac{f_X(x)}{f_Y(x)} = \left(\frac{\theta}{x}\right)^{\alpha_2 - \alpha_1} \frac{\Gamma \alpha_2 [1 + x(\alpha_2 - 1)]}{\Gamma \alpha_1 [1 + x(\alpha_1 - 1)]},$$

which is increasing in x for $\alpha_2 \leq \alpha_1$. Clearly, for $\theta_1 = \theta_2 = \theta$, X is stochastically greater than Y with respect to likelihood ratio if $\alpha_2 \leq \alpha_1$.

2.5. Renyi Entropy

An entropy is a measure of variation of the uncertainty in the distribution of a random variable X . For a probability distribution, the expression of the Renyi entropy (Renyi (1961)) is defined by,

$$e(\gamma) = \frac{1}{1 - \gamma} \log \int f^\gamma(x) dx. \quad (7)$$

where, $\gamma > 0$ and $\gamma \neq 1$. Substituting (4) in (7) and after some simplifications, we have

$$e(\gamma) = \frac{1}{1 - \gamma} \log \left\{ \frac{\theta^{(\alpha+1)(1-\gamma)} \Gamma(\gamma(\alpha+1) - 2) (\gamma\alpha + \gamma - 2 + \alpha\theta\gamma - \theta\gamma)}{(1 + \theta) \Gamma(\alpha) \gamma^{\gamma(\alpha+1)-1}} \right\}. \quad (8)$$

3. Maximum likelihood estimation

Suppose $X_1, X_2, X_3, \dots, X_n$ are independently and identically distributed random variables of size n from the extended inverse Lindley distribution. The likelihood function based on observed sample $\underline{X} = \{x_1, x_2, x_3, \dots, x_n\}$, is defined as

$$L = \frac{\theta^{n\alpha}}{(1 + \theta)^n (\Gamma\alpha)^n} \prod_{i=1}^n \frac{(1 + x_i(\alpha - 1))}{x_i^{\alpha+1}} e^{-\sum_{i=1}^n \frac{\theta}{x_i}}. \quad (9)$$

The log-likelihood function corresponding to (9) is given by

$$\log L = n\alpha \log \theta - n \log(1 + \theta) - n \log(\Gamma\alpha) + \sum_{i=1}^n \log(1 + x_i(\alpha - 1)) - (\alpha + 1) \sum_{i=1}^n \log(x_i) - \sum_{i=1}^n \frac{\theta}{x_i}. \tag{10}$$

The maximum likelihood estimates (MLEs) $\hat{\alpha}$ and $\hat{\theta}$ of α and θ , respectively can be obtained as the simulation solution of the following log-likelihood equations,

$$\frac{\delta \log L}{\delta \alpha} = n \log \theta - n(\alpha) + \sum_{i=1}^n \frac{x_i}{(1 + x_i(\alpha - 1))} - \sum_{i=1}^n \log x_i = 0, \tag{11}$$

$$\frac{\delta \log L}{\delta \theta} = \frac{n\alpha}{\theta} - \frac{n}{(1 + \theta)} - \sum_{i=1}^n \frac{1}{x_i} = 0, \tag{12}$$

where $(\alpha) = \frac{\delta \log \Gamma \alpha}{\delta \alpha}$. Using equation (12), the MLE of θ can be obtained in terms of α as

$$\hat{\theta}(\alpha) = \frac{-(S - \alpha + 1) + \sqrt{(S - \alpha + 1)^2 + 4S\alpha}}{2S}$$

where, $S = \frac{1}{n} \sum_{i=1}^n \frac{1}{x_i}$.

The MLE of α can be uniquely determined by solving the following non-linear equation,

$$n \log \hat{\theta}(\alpha) - n\zeta(\alpha) + \sum_{i=1}^n \frac{x_i}{(1 + x_i(\alpha - 1))} - \sum_{i=1}^n \log x_i = 0. \tag{13}$$

Note that the equation (13) cannot be solved analytically. Therefore, we need numerical iteration technique to solve the equation. Here, we use Newton-Raphson Method for solving the non-linear equations.

3.1. Approximate confidence intervals

Obtaining the exact confidence intervals for α and θ is not an easy task since the MLE's of α and θ are not in closed forms. We, therefore, can use the asymptotic behaviour of the maximum likelihood estimator to obtain the asymptotic confidence intervals for the parameters.

Using large sample theory of MLE, the asymptotic sampling distribution of $(\hat{\alpha}, \hat{\theta})$ is $N_2(0, \Delta^{-1})$

where $(\hat{\alpha}, \hat{\theta})$ are the MLEs of (α, θ) , and Δ is the observed Fisher's information matrix. The elements of Δ are given by

$$\Delta_{11} = \left| -\frac{\delta^2 \log L}{\delta \alpha^2} \right|_{\alpha=\hat{\alpha}}, \Delta_{12} = \Delta_{21} = \left| -\frac{\delta^2 \log L}{\delta \alpha \delta \theta} \right|_{\alpha=\hat{\alpha}, \theta=\hat{\theta}} \quad \text{and} \quad \Delta_{22} = \left| -\frac{\delta^2 \log L}{\delta \theta^2} \right|_{\theta=\hat{\theta}}, \text{ where, } \frac{\delta^2 \log L}{\delta \alpha} = n \log(\alpha) - \sum_{i=1}^n \frac{x_i}{[1+x_i(\alpha-1)]^2}, \frac{\delta^2 \log L}{\delta \alpha \delta \theta} = \frac{n}{\theta}, \frac{\delta^2 \log L}{\delta \theta} = \frac{-n\alpha}{\theta} + \frac{n}{(1+\theta)^2}.$$

The estimated variance covariance matrix of the parameters α and θ can be calculated by

$$\text{inverting } \Delta \text{ as follows } V^{-1} = \begin{Bmatrix} \widehat{\text{var}}(\alpha) & \widehat{\text{cov}}(\alpha, \theta) \\ \widehat{\text{cov}}(\theta, \alpha) & \widehat{\text{var}}(\theta) \end{Bmatrix} = \begin{Bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{Bmatrix}^{-1}.$$

The diagonal elements $\widehat{\text{var}}(\alpha)$ and $\widehat{\text{var}}(\theta)$ of this matrix are the asymptotic variances of the estimators of α and θ , respectively. Thus, the asymptotic $100(1 - \alpha)\%$ confidence intervals for α and θ are given by

$$\left\{ \begin{array}{l} \hat{\alpha}_L = \hat{\alpha} - Z_{\psi/2} \sqrt{\widehat{\text{var}}(\alpha)}, \quad \hat{\alpha}_U = \hat{\alpha} + Z_{\psi/2} \sqrt{\widehat{\text{var}}(\alpha)} \\ \hat{\theta}_L = \hat{\theta} - Z_{\psi/2} \sqrt{\widehat{\text{var}}(\theta)}, \quad \hat{\theta}_U = \hat{\theta} + Z_{\psi/2} \sqrt{\widehat{\text{var}}(\theta)} \end{array} \right\}$$

where, $Z_{\psi/2}$ is the upper $100 \times (\psi/2)^{th}$ percentile of a standard normal distribution.

4. Simulation Study

In this section, we investigate the properties of the MLEs of α and θ with respect to sample size n . For this purpose, we need to simulate the random sample from the EILD. Since the EILD is two component mixture of inverse gamma distributions, we can use the definition of the mixture distribution for generating the random numbers from the distribution. Algorithm for sample generation from the EILD is given by

Algorithm 1:

- Step 1. Set $n, \alpha, \& \theta$.
- Step 2. Generate an inverse gamma random variable, $V_j \sim IG(\alpha - 1, \theta)$ for $j = 1, 2, \dots, n$.
- Step 3. Generate an inverse gamma random variable, $W_j \sim IG(\alpha, \theta)$ for $j = 1, 2, \dots, n$.
- Step 4. Generate a Uniform random variable, $U_j \sim U(0, 1)$ for $j = 1, 2, \dots, n$.
- Step 5. If $U_j \leq (\theta/(\theta + 1))$, then set $X_i = V_i$, otherwise, set $X_i = W_i$ for $j = 1, 2, \dots, n$.

On the basis of simulated samples, we study the behaviour of biases, absolute biases and mean squared error (MSE) of MLEs of α and θ with varying sample size n . We also investigated the coverage probability for the asymptotic confidence intervals. Simulation algorithm consists the following steps:

Algorithm 2:

- Step 1. Generate 10,000 samples of size n from the EILD(2, 2) by using the Algorithm 1.
- Step 2. Compute the MLEs for the 10,000 samples, say $\hat{\alpha}_i, \hat{\theta}_i$ for $i = 1, 2, \dots, 10,000$.
- Step 3. Compute the standard errors of the MLEs for the 10,000 sample say $s_{\hat{\alpha}_i}, s_{\hat{\theta}_i}$. The standard errors were computed by inverting the observed information matrices.
- Step 4. Compute average estimate (AE) as

$$AE(\hat{\Theta}) = \frac{1}{n} \sum_{i=1}^n \hat{\Theta}_i$$

- Step 5. Compute the biases, absolute biases (AB) and mean squared error (MSE) using the following formulae,

$$B_{\Theta}(n) = \frac{1}{10,000} \sum_{i=1}^{10,000} (\hat{\Theta}_i - \Theta)$$

$$AB_{\Theta}(n) = \frac{1}{10,000} \sum_{i=1}^{10,000} |(\hat{\Theta}_i - \Theta)|$$

$$MSE_{\Theta}(n) = \frac{1}{10,000} \sum_{i=1}^{10,000} (\hat{\Theta}_i - \Theta)^2$$

for $\Theta = \{\alpha, \theta\}$.

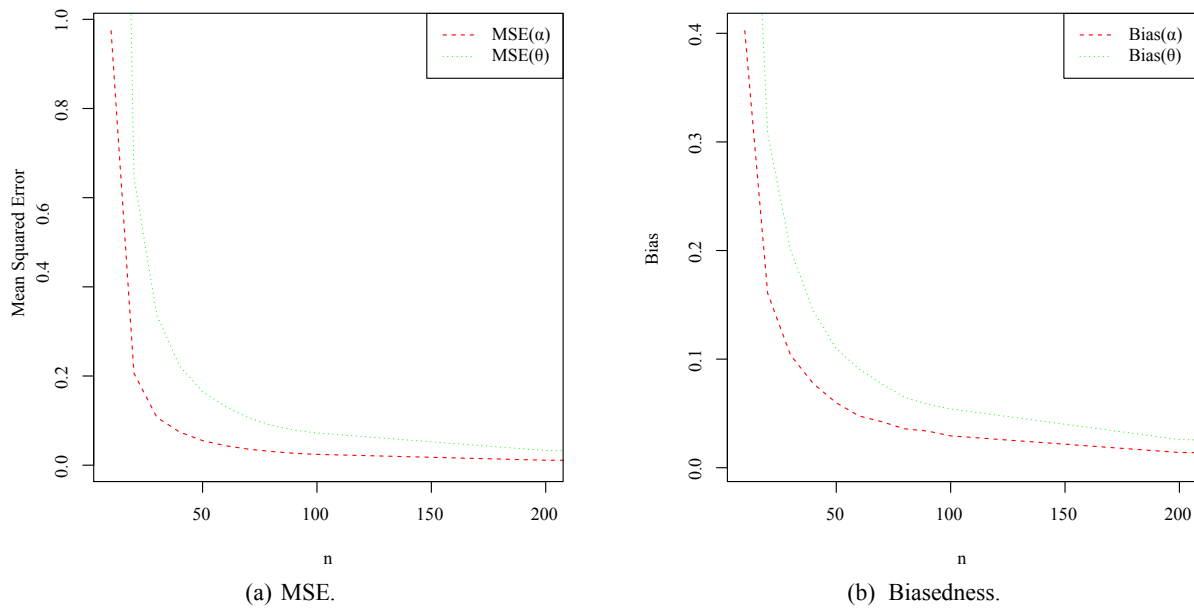


Figure 4: Mean squared error (MSE) and Biasedness of $\hat{\alpha}$ and $\hat{\theta}$ with varying n .

Step 6. Compute the coverage probabilities (CP) and coverage widths (CW) given by

$$CP_{\Theta}(n) = \frac{1}{10,000} \sum_{i=1}^{10,000} I \left(\hat{\Theta}_i - 1.959964s_{\hat{\Theta}_i} \leq \Theta \leq \hat{\Theta}_i + 1.959964s_{\hat{\Theta}_i} \right)$$

$$CW_{\Theta}(n) = \frac{3.919928}{10,000} \sum_{i=1}^{10,000} s_{\hat{\Theta}_i}$$

for $\Theta = \{\alpha, \theta\}$, where $I(\cdot)$ denotes the indicator function.

We repeat these steps for $n = 10, 20, 30, 40, 50, 60, 70, 80, 90, 100, 200, 500, 1000$ with $\alpha = 2$ and $\theta = 2$, and computed $Bias_{\Theta}(n)$, $MSE_{\Theta}(n)$, $CP_{\Theta}(n)$ and $CW_{\Theta}(n)$. Trends for the results have also been shown graphically in Figures 4, 5, 6. From the results, it can be observed that the biases and MSEs of the MLEs of α and θ decrease as sample size n increases. That proves the consistency of the MLEs. In general, the CP for the CIs of α and θ decrease with increasing sample size. However, it stabilizes at approximately 0.95 for large sample size. The average width of CIs for α and θ decreases as sample size increase.

5. Real data analysis

In this section, we use a set of real data (given in table 1) of flood levels to demonstrate the applicability of the extended inverse Lindley distribution. The data were obtained in a civil engineering context and gives the maximum flood level (in millions of cubic feet per second) for the Susquehanna river at Harrisburg, Pennsylvania over 20 four-year periods from 1890 to 1969. These data have been widely discussed by many authors and were initially reported by Dumonceaux and Antle (1973). First, we use the likelihood ratio (LR) test statistic to check

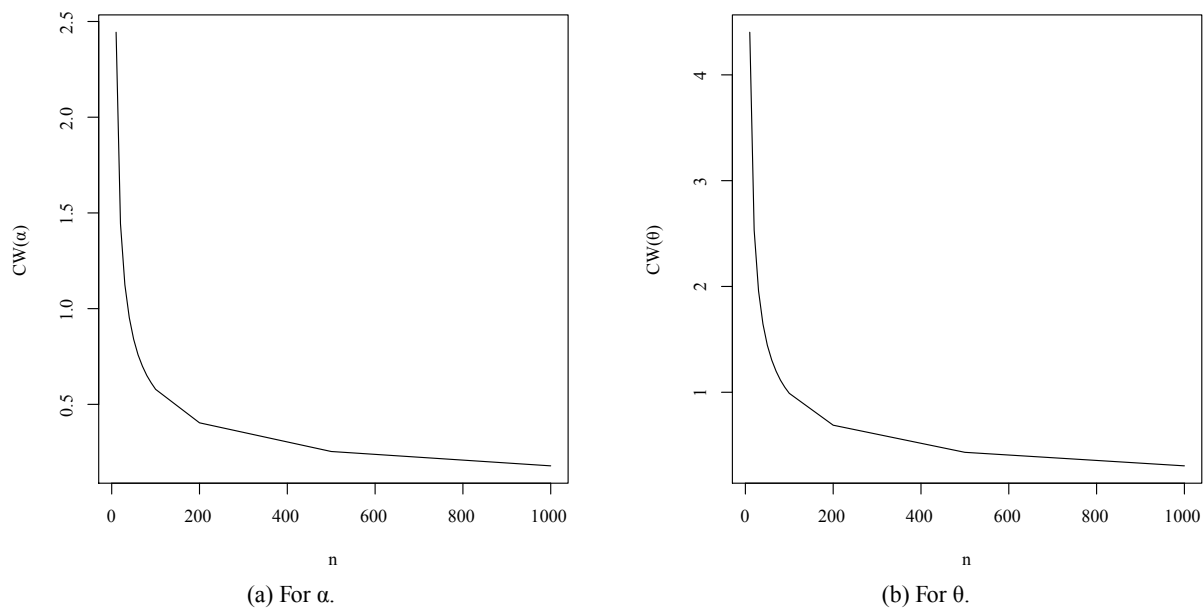


Figure 5: Coverage width of asymptotic confidence intervals of $\hat{\alpha}$ and $\hat{\theta}$ with varying n .

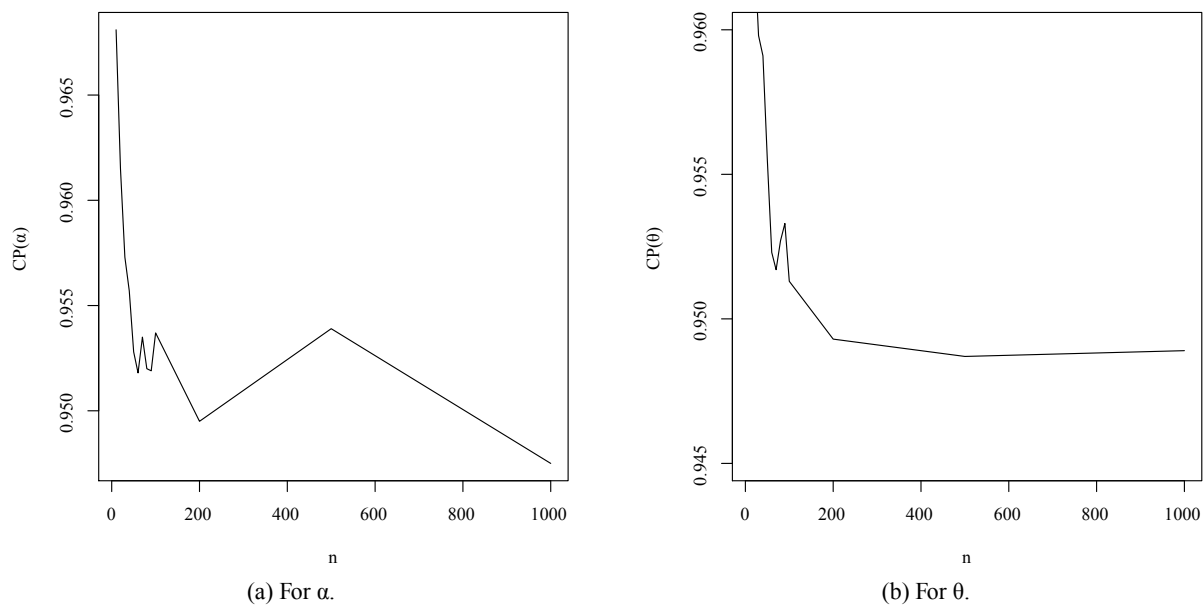


Figure 6: Coverage probability of asymptotic confidence intervals of $\hat{\alpha}$ and $\hat{\theta}$ with varying n . Table 1:

Flood level data

0.654	0.654	0.315	0.449	0.297
0.324	0.269	0.74	0.418	0.412
0.338	0.392	0.484	0.265	0.379
0.423	0.379	0.402	0.494	0.416

whether the shape parameter associated with extended inverse Lindley distribution improves its applicability. The hypothesis can be stated as

$$\begin{aligned} &\text{null hypothesis; } H_0 : X \sim \text{ILD}(\theta) \text{ (i.e. } \alpha = 1) \\ &\qquad\qquad\qquad \text{versus} \\ &\text{alternative hypothesis; } H_1 : X \sim \text{EILD}(\alpha, \theta) \text{ (i.e. } \alpha \neq 1). \end{aligned}$$

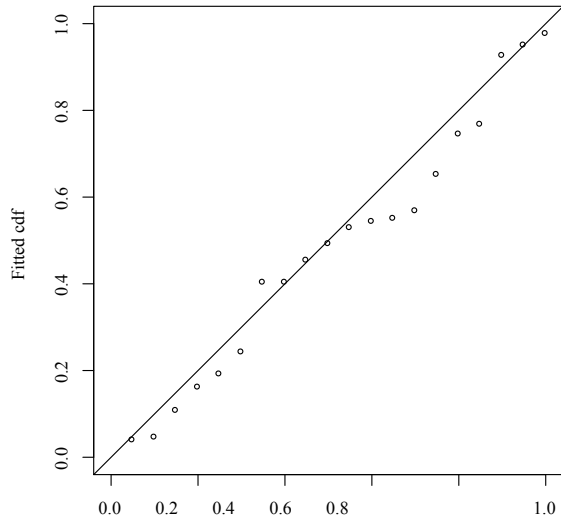
In this case, the LR test statistic for testing H_0 versus H_1 is $\zeta(l_0 - l_1)$, where l_1 and l_0 are the log-likelihood functions under H_1 and H_0 , respectively. The statistic ζ is asymptotically (as $n \rightarrow \infty$) distributed as χ_k^2 with k degree of freedom, where k is the number of parameters. The LR test rejects H_0 if $\zeta \geq \chi_k^2(\gamma)$, where $\chi_k^2(\gamma)$ denotes the upper $100\gamma\%$ quantile of the χ_k^2 distribution. For given real data set, the log-likelihood under the ILD is $l_0 = 0.5854$ with $\hat{\theta} = 0.6344$ and under H_1 , l_1 is 16.1422 with ($\hat{\alpha} = 15.556, \hat{\theta} = 5.7875$). Clearly, the shape parameter α can never be 1 for the data since $\hat{\alpha} = 15.556$ which is very far than unity. However, the LR test statistic is $\zeta = 31.11$ which is greater than $\chi_1^2(0.05) = 3.84$. The results indicate that evidences do not support the null hypothesis. Therefore, the EILD is a better model than its special case, ILD.

Further, we also applied another statistical tools such as Kolmogorov-Smirnov (KS) statistic, Akaike information criterion (AIC) and Bayesian information criterion (BIC), to choose the best possible model for the data set among all competitive models. For this purpose, we consider seven alternative distribution such as inverse Weibull ($\text{IW}(\alpha, \theta)$), inverse gamma ($\text{IGM}(\alpha, \theta)$), generalised inverse exponential ($\text{GIE}(\alpha, \theta)$), inverse Gaussian ($\text{IG}(\alpha, \theta)$), lognormal ($\text{LN}(\mu, \sigma^2)$), log-Logistic ($\text{LLog}(\alpha, \beta)$) and flexible Weibull ($\text{FW}(\alpha, \theta)$). The statistical tools used are described as follows: KS test statistic, $D_n = \sup_x |F(x) - F_n(x)|$, $\text{AIC} = -2 \log L + 2p$, and $\text{BIC} = -2 \log L + p \log n$, where, p is the number of parameters are to be estimated from the data.

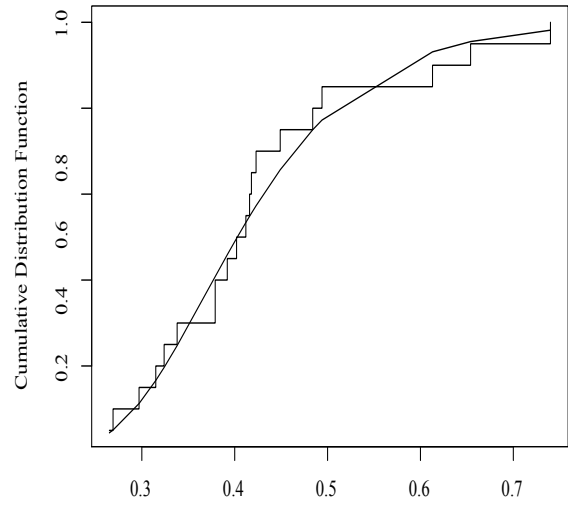
The selection criterion is that the lowest AIC and BIC correspond to the best model fitted. The MLEs, AIC and BIC are shown in Table 2. From the Table, we can observed that the extended inverse Lindley distribution shows the smaller AIC and BIC than other competing distributions. Thus, the EILD fits well the data set. The plots of probability-probability and the fitted cumulative distribution of the EILD are shown in Figure 7 for maximum flood level data. Figures also indicate that the EILD is a good fitted model for the data. The fitted density of EILD is shown in Figure 8.

Table 2: MLEs, AIC, BIC and KS for the fitted models based on real data.

Model	MLE	LogL	AIC	BIC	KS
EILD($\hat{\alpha}, \hat{\theta}$)	$\hat{\alpha} = 15.556, \hat{\theta} = 5.7875$	16.1422	-32.2845	-26.2930	0.1271
IW($\hat{\alpha}, \hat{\theta}$)	$\hat{\alpha} = 4.3140, \hat{\theta} = 0.0119$	16.0974	-32.1947	-26.2033	0.1060
IGM($\hat{\alpha}, \hat{\theta}$)	$\hat{\alpha} = 14.578, \hat{\theta} = 5.7383$	16.1416	-32.2833	-26.2918	0.1272
GIE($\hat{\alpha}, \hat{\theta}$)	$\hat{\alpha} = 37.7001, \hat{\theta} = 1.645$	15.4112	-30.8224	-24.8309	0.1584
IG($\hat{\alpha}, \hat{\theta}$)	$\hat{\alpha} = 0.4231, \hat{\theta} = 5.6410$	15.8520	-31.7040	-25.7126	0.1469
LN($\hat{\mu}, \hat{\sigma}^2$)	$\hat{\mu} = -0.897, \hat{\sigma}^2 = 0.269$	15.8242	-31.6484	-25.6570	0.2843
LLog($\hat{\alpha}, \hat{\beta}$)	$\hat{\alpha} = 0.4018, \hat{\beta} = 6.5752$	15.6859	-31.3717	-25.3803	0.9999
FW($\hat{\alpha}, \hat{\theta}$)	$\hat{\alpha} = 3.6456, \hat{\theta} = 0.8217$	13.5886	-27.1773	-21.1858	0.3149



Probabilit y-probability (PP) plot.



Cumulativ e distribution function.

Figure 7: Probability-probability (PP) and fitted cdf of EILD based on real data.

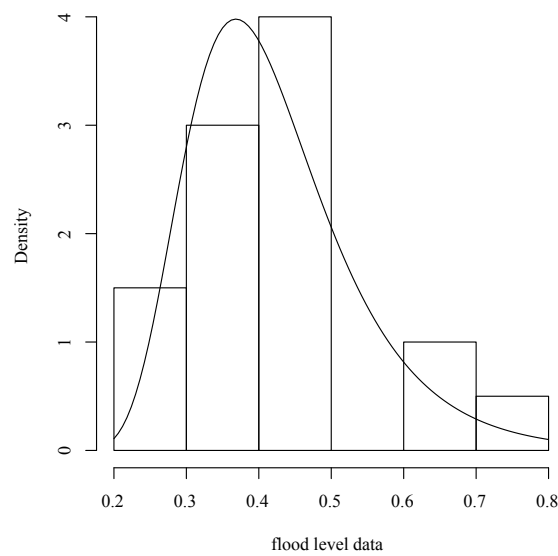


Figure 8: Plot of the fitted density of the EILD for real data.

6. Summary and conclusion

In this article, we introduced two parameter extension of inverse Lindley distribution, called extended inverse Lindley distribution (EILD). The distribution shows the upside-down bathtub shape for its hazard rate. The current extension provides a rather general and flexible framework for statistical analysis of positive data. The EILD can be expressed as a two component mixture of the inverse gamma distributions. That provides some expansions for the ordinary moments, mean deviations, stochastic ordering, inverse moments and Renyi entropy. The estimation of parameters is approached by the method of maximum likelihood. The confidence intervals for the parameters of EILD are also obtained by using asymptotic distribution of MLEs. Simulation study has been carried out to study the behaviours of MLEs of the parameters with respect to sample size. We considered the likelihood ratio test statistic to compare the model with its baseline model. An application of the EILD shows that it could provide a better fit than other alternative inverse distributions. Finally, we hope that our proposed model will attain great applicability in the real problems encountered in various disciplines such as medical, engineering and social sciences etc.

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